Delhi Institute for Administrative Services
India's Leading Institute for Civil Services Examination

CIVIL SERVICES EXAMINATION (MAINS) 2020
DIAS PHYSICS OPTIONAL TEST SERIES
Test 9: FULL SYLLABUS TEST PAPER - I

Time Allowed: 3 Hours **Max Marks: 250**

All Questions are compulsory .

Q1. (a) The density inside a solid sphere of radius a is given by $\rho = \frac{\rho_0 a}{r}$ where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centres

(b) Prove mathematically that the addition of any velocity of a particle to the velocity of light in free space merely reproduces the velocity of light in free space only.

(c) How does one obtain the angular velocity of the Earth about the North Pole with respect to a fixed star as 7.292×10^{-5} sec^{-1} ? Explain your method of calculating the above value

(d) What is the physical significance of Einstein's A-coefficient ? Explain why it is more difficult to achieve Lasing action at X-ray wavelength than at infra-red wavelength.

(e) For a multimode step index optical fibre, the core refractive index is 1.5 and fractional index difference is 0.001 . Calculate the pulse broadening for 1 km length of the fibre. Over a length of 2 km of the fibre, calculate the minimum pulse separation that can be transmitted without overlap

Q2. (a) Consider a rigid body rotating about an axis passing through a fixed point in the body with an angular velocity $\vec{\omega}$. Determine the kinetic energy of such a rotating body in a coordinate system of principal axis. If the Earth suddenly stops rotating, what will happen to the rotational kinetic energy? Comment in detail.

(b) A body turns about a fixed point. Show that the angle between its angular velocity vector and its angular momentum vector about a fixed
point is always acute.

(c) Explain the working principle of a 3-level laser with a specific example.
Comment on why the third level is needed.

Q3. (a) A mirror is moving through vacuum with a relativistic speed v in the x-direction. A beam of light with frequency ω_i is normally incident (from $x = \infty$) on the mirror.

(i) What is the frequency of the reflected light expressed in terms of ω_i, c and v?

(ii) What is the energy of each reflected photon?

(b) In question 3(a), if the average energy flux of the incident beam is $P_i\left(\frac{\text{watts}}{m^2}\right)$ what is the average energy flux of the reflected beam?

(c) In a certain engine, a piston undergoes vertical SHM with an amplitude of 10 cm. A washer rests on the on the top of the piston. As the motor is slowly speeded up at what frequency will the washer no longer stay in contact with the piston?

Q4. (a) Discuss the problem of scattering of charged particle by a coulomb field. Hence, obtain an expression for Rutherford scattering cross-section. What is the importance of the above expression?

(b) A charged particle is moving under the influence of a point nucleus. Show that the orbit of the particle is an ellipse. Find out the time period of the motion.

(c) Considering a plane transmission diffraction grating, where d is the distance between two consecutive ruled lines, m as the order number and 0 as the angle of diffraction for normal incidence calculate the angular dispersion $\frac{d\theta}{d\lambda}$ for an incident light of wavelength λ,

SECTION B

Q5. (a) In a Young double slit experiment, the first bright maximum is displaced by $y=2$ from the central maximum. If the spacing between slits and distance from the screen are 0 0.1 mm and 1 m respectively, find the wavelength of light.

(b) Define Enthalpy and show that it remains constant in a throttling process.

(c) How does holography differ from conventional photography? What are the requirements for the formation and reading of a hologram?

(d) In deriving radiation laws, we consider a cubical container of volume V containing a photon gas in equilibrium. Calculate the differential number of allowed normal modes of frequency ω

(e) Starting from Maxwell's equation, obtain the wave equation for the electric field $\vec{E}$ in free space and appropriate wave equation for the electric field $\vec{E} = E_z(x,y,z)\hat{z}$.

Q6. (a) Show that the group velocity is equal to particle velocity. Also prove that group velocity of the photons is equal to c, the velocity of light.

(b) For initial current conditions $I = I_0$ and $\frac{dI}{dt} = 0$ at $t = 0$, show that the time dependent current in the critical damping case for an LCR circuit is given by

$$I = I_0\left(1 + \frac{\gamma t}{2}\right)e^{\frac{-\gamma t}{2}}$$

where $\gamma = \frac{R}{L}, \omega_0^2 = \frac{1}{LC}, \omega\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ and $\tan\delta = \frac{-\gamma}{2\omega}$

(c) Using Ampere's Law and continuity equation, show that the divergence of the total current density is zero.

Q7. (a) State and explain Stefan-Boltzmann Law. Show that $\log P = \log K + 4\log R$, where P is the power emitted by black body and R is the resistance of the black body, K is a constant.

(a)

(b) One kg of water at 20^0C is converted into, ice at -10^0C at constant pressure. Heat capacity of water is $4,200\ J/\lg.K$ and that 'of ice is $2,100\ J.kg.K$. Heat of fusion of ice at 0^0C is $335\times10^3\ J/kg$. Calculate the total change in entropy of the system.

(c) When connected in series, L_1, C_1 have the same resonant frequency as L_2, C_2 also connected in series. Prove that if all these circuit elements are connected in series, the new circuit will have the same resonant frequency as either of the circuits first mentioned.

(d) Show that the energy flow due to a plane electromagnetic wave propagating along z-direction in a dielectric medium is given by

$$\hat{z}\frac{k}{\omega\mu}E_0^2\cos^2(kz - \omega t)$$

where k and ω are the propagation vector and angular frequency, E_0 is electric field amplitude, μ is the relative permeability of the medium.

Q8. (a) Consider a system of free gas particles having f degrees of freedom. Use equipartition theorem to establish the relation

$$f = \frac{2}{\left(\frac{C_P}{C_V} - 1\right)}$$

where C_P and C_V are molar specific heats at constant pressure and constant volume respectively. Obtain the values of $\frac{C_P}{C_V}$ for diatomic and triatomic gases.

(b) Show that both Fermi-Dirac and Bose-Einstein distribution functions at an energy E are given by:

$$f(E) \simeq \exp\left[\frac{(\mu - E)}{k_B T}\right],$$

where $f(E)$ is much smaller than unity, μ. and $k_B T$ are the chemical potential and thermal energy of the atom.

(c) Explain the four thermodynamic relations of Maxwell. Using the same, obtain the Clausius-Clapeyron equation

$$\frac{dP}{dT} = \frac{L}{T(V_2 - V_1)}$$

(d) Using Maxwell-Boltzmann distribution law prove that there cannot be any negative absolute temperature.

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Name - Mamta Yadav

Roll No. - 0829179

Date - 20th Dec '2019

Time - 5:00pm - 8:00pm

Paper I - full Length Test -9

Q1	Q5	Q4	Q7	Q8
$17\frac{1}{2}$	$29\frac{1}{2}$	31	30	25

$\frac{133}{250}$ Good Presentation

Need to be careful in Q1 & Q5

Q4. (a) The problem of scattering of charged particle by a coulomb field is essentially a problem of elastic collision between the charged particle and the Nucleus. The following assumptions are made

→ The force acting between the Charged Particle and the Nucleus is of Electrostatic Coulombic Repulsion $\boxed{F = q_1 q_2 / 4\pi\varepsilon_0 r^2}$

→ The Nucleus is much heavier and hence considered at rest.

→ The Trajectory ~~is~~ of the Charged Particle is hyperbolic as the force is repulsive in nature.

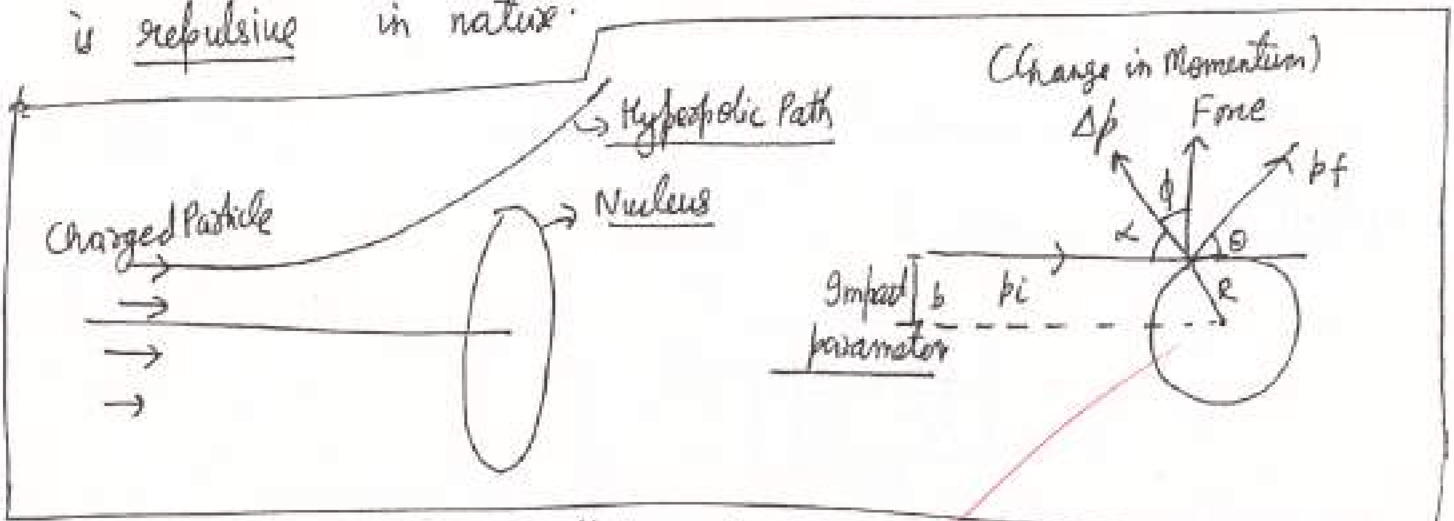

Fig. Scattering of Charged Particle by Coulomb field

Rutherford Scattering Cross Section

Differential Scattering Cross section can be defined as the ratio of Incident Flux of Particles (I_0) to the Number of Particles scattered per unit solid angle per unit time ($I(\theta)$).

$$\boxed{\sigma(\theta) = \frac{I_0}{I(\theta)} = \frac{-b}{\sin\theta}\frac{db}{d\theta}} \quad — \text{(1)}$$

Here b is the Impact Parameter.

From the figure, $\Delta p_x = p\cos\theta - p$

$\Delta p_y = p\sin\theta$

$$\Delta p = \sqrt{(\Delta p_x)^2 + (\Delta p_y)^2} = \sqrt{p^2(\cos\theta - 1)^2 + p^2\sin^2\theta}$$

$$\boxed{\Delta p = 2p\sin\frac{\theta}{2}}$$

This change in momentum is due to the electrostatic Coulomb Force which acts for a small time as an Impulse.

$$\Delta p = \int F\cos\phi\, dt$$

$$2p\sin\frac{\theta}{2} = \int_{-\alpha}^{+\alpha} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r^2}\cos\phi\, dt \quad \text{—(2)}$$

from the Law of Conservation of Angular Momentum (as the force is a central force), $mv_0 b = mr^2\frac{d\phi}{dt}$

$$\boxed{\frac{dt}{r^2} = \frac{d\phi}{v_0 b}} \quad \text{—(3)}$$

using (3) in (2), $2p\sin\frac{\theta}{2} = \int_{-\alpha}^{+\alpha} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\cos\phi\frac{d\phi}{v_0 b}$

Impact Parameter, $b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0(2E_\alpha)}\frac{\sin\alpha}{\sin\frac{\theta}{2}}$

from the figure, $2\alpha + \theta = 180^\circ \Rightarrow \alpha = 90 - \frac{\theta}{2} \Rightarrow \sin\alpha = \cos\frac{\theta}{2}$

$$\boxed{b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0(2E_\alpha)}\cot\frac{\theta}{2}}$$

using this in (1)

Rutherford Scattering Cross-section $\sigma(\theta) = -\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0(2E_\alpha)} \frac{\cot\theta/2}{\sin\theta} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0(2E_\alpha)} - \operatorname{cosec}^2\frac{\theta}{2}$.

$$\boxed{\sigma(\theta) = \frac{Z_1^2 Z_2^2 e^4}{(4\pi\epsilon_0)^2 16E_\alpha^2} \operatorname{cosec}^4\left(\frac{\theta}{2}\right)}$$

This is the Required expression

Importance of the Above formula

→ It gives us the total cross-section, ie, the area of influence.

$$\sigma = \int \sigma(\theta)\, d\Omega$$

$12\frac{1}{2}/20$

→ The Number of Particles deflected in a direction is directly proportional to $\operatorname{cosec}^4\left(\frac{\theta}{2}\right)$. Therefore, greater the deflection, lesser will be the number of particles.

b) Let the charged particle be orbiting be moving under the influence of a point nucleus in an arbitrary motion. The two body problem can be reduced into a one body problem using reduced mass $\mu = \frac{m_1 m_2}{m_1 m_2}$

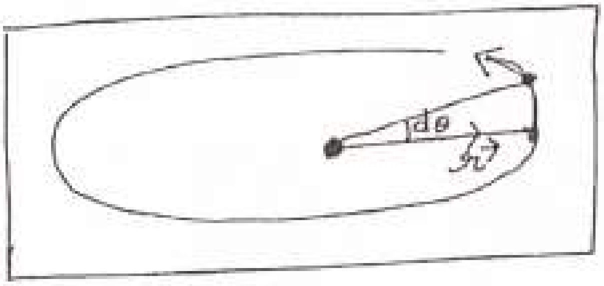

Fig. Motion of Charged Particle

The Coulombic force is a Central force. Hence, the Torque is zero & the Angular momentum is constant. $\boxed{J = mr^2 \frac{d\theta}{dt} = \text{Constant}}$

$$F = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

Let $u = \frac{1}{r}$. Then, $\frac{dr}{dt} = -\frac{J\,du}{m\,d\theta}$ and $\frac{d^2r}{dt^2} = -\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2}$ — ②

using ① & ② in (F) above,

$$F = m\left[-\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{1}{u}\frac{J^2u^4}{m^2}\right]$$

$$\boxed{F = -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right)}$$

Let $F = -\frac{k}{r^2} = -ku^2$, Then $\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mk}{J^2}}$

This is the general equation for the trajectory which gives the solution, $u = A\cos\theta + \frac{mk}{J^2}$

$$\boxed{\frac{J^2u}{mk} = 1 + \frac{AJ^2}{mk}\cos\theta}$$

Comparing with the equation of conic section, eccentricity $e = \frac{AJ^2}{mk}$

& $l = \frac{J^2}{mk}$

Kinetic Energy, $K = E - U = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right)$

~~2(E+~~ $$E + ku = \frac{m}{2}\left(\frac{J^2}{m^2}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}\frac{J^2u^4}{m^2}\right)$$

$$E + ku = \frac{J^2}{2m}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$$

using ④ from above, we get,

$$A = \frac{mk}{J^2}\left(1+\frac{2EJ^2}{mk^2}\right)^{1/2}$$

Eccentricity, $\boxed{e = \sqrt{1+\frac{2EJ^2}{mk^2}}}$

for Attractive force, $E<0$. Therefore, eccentricity $e<1$ or the Trajectory is elliptical.

Time period

$$A = \int_0^T \left(\frac{dA}{dt}\right) dt \quad \text{where } \frac{dA}{dt} = \text{areal velocity} \Rightarrow \frac{dA}{dt} = r^2\frac{d\theta}{dt} = \frac{J}{2m} \text{ (const)}$$

$$A = \int_0^T \frac{J}{2m}\, dt = \frac{JT}{2m}$$

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$$A = \text{area of ellipse} = \pi ab = \frac{JT}{2m}$$

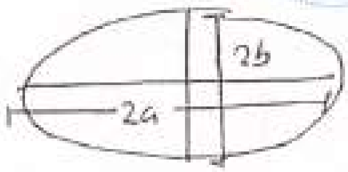

Fig. ellipse

Therefore, $\boxed{T = \frac{2m\pi ab}{J}}$

we know, $b^2 = a^2(1-e^2)$ & $\frac{l}{r} = 1+e\cos\theta$

This gives $\frac{a}{l} = \frac{1}{1-e^2}$

$$T = \frac{2m\pi a^2(1-e^2)^{1/2}}{J} = \frac{2\pi ma^2}{J^2}\left(\frac{l}{a}\right)^{1/2}$$

$$\boxed{T = \frac{2\pi m l^{1/2} a^{3/2}}{J}} \Rightarrow \text{Time period } \boxed{T \propto a^{3/2}}$$

(Kepler's Third Law)

This is the required time period of the Motion.

(c) Diffraction Grating can be defined as an optical device which comprises of numerous slits separated by opaque portions. Light is transmitted from slits & blocked by opaque portions giving numerous phenomenon.

Equation of Diffraction Grating

$$\boxed{(a+b)\sin\theta_m = m\lambda} \quad -①$$

Here $a+b=d$ is the distance b/w 2 consecutively ruled lines.

Angular Dispersion can be defined as the angular difference ($d\theta$) between two different different wavelength (λ & $\lambda+d\lambda$)

$$\boxed{D = \frac{d\theta}{d\lambda}} \quad -②$$

from ①, $(a+b)\cos\theta_m \frac{d\theta_m}{d\lambda} = m$

$$\frac{d\theta_m}{d\lambda} = \frac{m}{(a+b)\cos\theta_n}$$

Therefore, from ②, $\boxed{D = \frac{m}{(a+b)\cos\theta_n}}$

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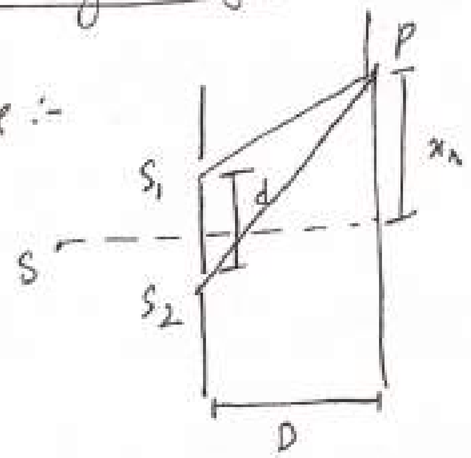
Angular Dispersion depends on → m (Order of Diffraction)

→ $\frac{1}{a+b}$, ie, Number of lines per unit distance.

This is contrary to ~~ang~~ resolving power which depends on total number of lines in the grating

Q5. (a) Young Double Slit Experiment gives Interference Pattern by the method of Division of Wavefront.



The Location of nth Bright/maxima fringe :-

$$\boxed{x_n = \frac{n\lambda D}{d}}$$

Given: $n = 1$

$x_n = 2mm = 2\times10^{-3} m$

D = Distance between screen & slit = 1

d = Distance between slits = 0.01 mm = 0.01×10^{-3} m

$$2\times10^{-3} = \frac{1\times\lambda\times1m}{0.01\times10^{-3}m}$$

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$$\boxed{\text{Wavelength, } \lambda = 2\times10^{-8} m}$$

Here we can taken the approximation $\boxed{D>>d}$ which gives straight fringes. Otherwise the fringes are Hyperbolic in pattern.

(b) Enthalpy (H) is a state variable. It is a thermodynamic potential which completely defines the state of an Adiabatic and Isobaric system. In such a system, change is always in the direction of reduced enthalpy, i.e. $\boxed{dH \leq 0}$.

$$\boxed{\text{Enthalpy, } H = U + pV}$$

Throttling Process – In a throttling process, a gas is made to expand from High Pressure (p_1) to Low Pressure ($p_2 < p_1$) through a porous plug. The process takes place without exchange of heat (i.e. Adiabatic Process). Hence, there is reduction in temperature on account of work done against intermolecular forces in porous plug.

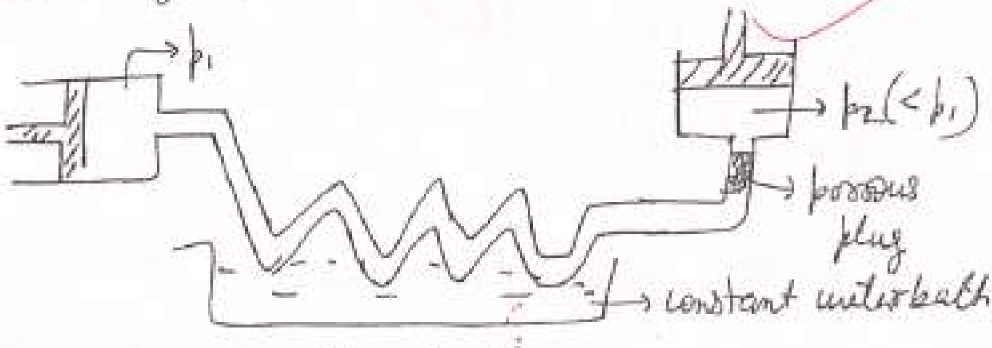

Fig. Throttling process

from Ist Law of Thermodynamics, $\delta Q = dU + pdV$

For Adiabatic Process, $0 = U_2 - U_1 + p_2V_2 - p_1V_1$

$$U_1 + p_1V_1 = U_2 + p_2V_2$$

$$\boxed{H_1 = H_2}$$

Therefore, Enthalpy is constant in a throttling process.

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(c) Holography is a technique to record 3D-Images of Objects

Holography	Conventional Photography
→ Both Intensity and Amplitude variation is recorded of the object.	→ Only Intensity Distribution of the Object is recorded.
→ It creates 3D-Images.	→ It creates 2D-Images.
→ ~~Only~~ Even small piece of the recorded film can produce the entire image	→ Picture can be easily damaged physically.

Holography has two processes ⇒ Recording the Hologram
⇒ Reconstruction of the Image

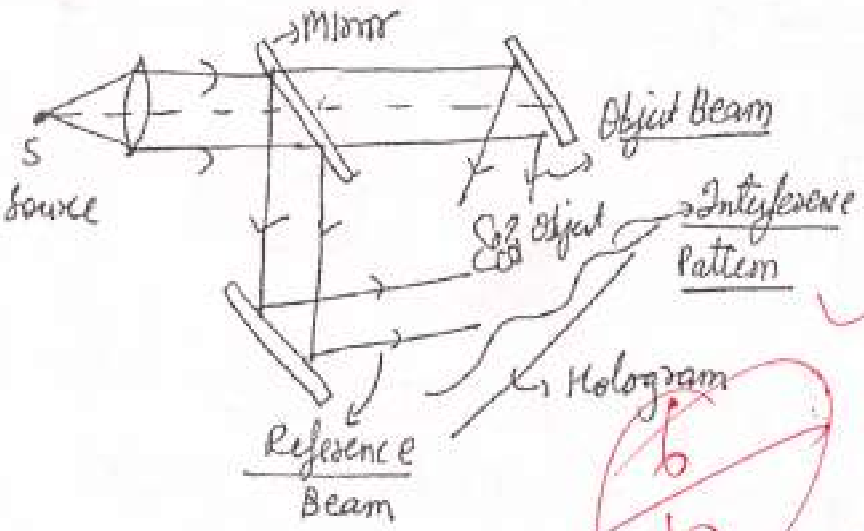

Fig. Recording of the Hologram

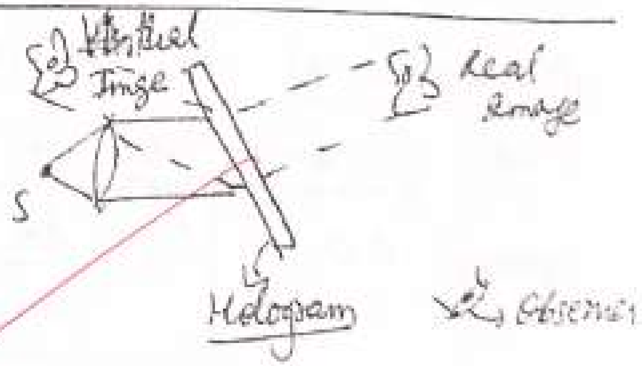

Fig. Reconstruction of Image

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Requirements → Source should be LASER for Monochromaticity, Coherence, Intensity, Directionality

→ The Length of the source should be such as to ensure Temporal & Spatial Coherence.

→ The Path Difference between ~~Inte~~ Object Beam & Reference Beam should be less than coherence length to ensure Interference.

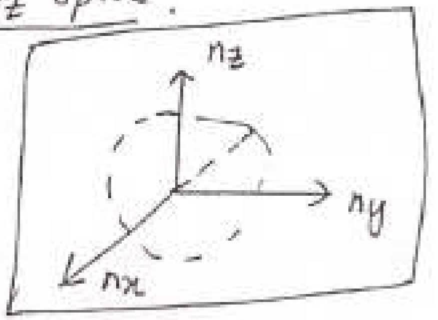
d) The Radiation is the container undergoes multiple reflections which results in formation of Standing Waves.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\psi(x,y,z,t)=A\exp(i\omega t)\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_z\pi z}{L}\right)$$

This where, $n_x^2+n_y^2+n_z^2=\left(\frac{2\nu L}{v}\right)^2=\left(\frac{\omega L}{\pi v}\right)^2$ — ①

This is the equation of Radius of sphere in n_x, n_y, n_z space.



Differential Number of allowed normal modes of frequency ω is the number of Modes in the frequency range ω to $\omega+d\omega$ per unit frequency $d\omega$.

$$\boxed{N=\frac{dn}{d\omega}}$$

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The number of Modes in the frequency range ω to $\omega+d\omega$ is the volume of spherical shell of radius R to $R+dR$ in the first quadrant.

$$dn=\frac{1}{8}\times 4\pi R^2 dR \quad \text{—②}$$

from ①, $R^2=\left(\frac{\omega L}{\pi V}\right)^2$ and $dR=\frac{L}{\pi V}d\omega$

$$dn=\frac{\omega^2 V\,d\omega}{2v^3\pi^2} \Rightarrow$$ Differential Number of Modes $\boxed{N=\frac{\omega^2 V}{2v^3\pi^2}}$ 2× $\left(\frac{4\pi V}{v^3}\right)\nu^2 d\nu$

Photons are e.m wave Hence 2 modes of Polarisation

e) The Maxwell's Equations for free space are :-

(1) $\vec{\nabla}\cdot\vec{E} = 0$ (2) $\vec{\nabla}\times\vec{E} = \frac{-\partial \vec{B}}{\partial t}$

(3) $\vec{\nabla}\cdot\vec{B} = 0$ (4) $\vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Taking curl of 2, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t}$

$$\underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{=0 \text{ from (1)}} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\nabla^2\vec{E} = \frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

This is the required wave equation for electric field. Comparing with wave equation, $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{4\pi\times10^{-7}\times8.85\times10^{-12}}}$

$$\boxed{v = 3\times10^8\, ms^{-1} = c}$$

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Therefore, EM waves travel with speed of light or light is an EM wave.

~~The solution of the wave equation, E = E₀ cos (kz−ωt)~~

If $\boxed{\vec{E} = E_z(x,y,z)\,\hat{z}}$, then wave propagating in either x-direction or y-direction.

If wave in x-axis $\boxed{\frac{\partial^2 E}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}}$

If wave is moving in y-axis $\boxed{\frac{\partial^2 E}{\partial y^2} = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}}$

(Electric field has component in z-axis but changes with x or y.

Q7. (a) According to the Stefan-Boltzmann's Law, the energy emitted by a body a temperature T per unit time per unit area is directly proportional to the fourth power of Temperature T.

$$\boxed{P \propto T^4 \Rightarrow P = \sigma T^4}$$

where σ = Stefan-Boltzmann Constant

$\sigma = 5.672 \times 10^{-7}\ W m^{-2} K^{-4}$

The Stefann-Boltzmann Law can be proved by the Planck's radiation law, according to it, Spectral energy Density, u_ν is

$$\boxed{u(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}}$$

$$u = \int_0^\infty u(\nu, T)\, d\nu = \int_0^\infty \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

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Let, $x = \frac{h\nu}{k_B T}$. then, $dx = \frac{h\, d\nu}{k_B T}$.

$$u = \frac{8\pi k_B^4 T^4}{h^3 c^3} \int_0^\infty \frac{x^3\, dx}{e^x - 1}$$

Integral:- $\int_0^\infty \frac{x^3\, dx}{e^x - 1} = \frac{\pi^4}{15}$

$$u = \left(\frac{8\pi^4 k_B^4}{15 h^3 c^3}\right) T^4$$

Power, $P = \frac{uc}{4}$

$$P = \left(\frac{2\pi^4 k_B^4}{15 h^3 c^2}\right) T^4$$

The expression in the bracket is a constant (say K).

Therefore, $P = KT^4$

$$\boxed{\log_e P = \log_e K + 4 \log_e T}$$

Here T = R is the Resistance of the Black body.

b) Entropy is a state variable which is a measure of ~~molecular~~ disorder in the molecular arrangement of the System.

Entropy, $\Delta S = \int \frac{\delta Q}{T}$

1 kg Water at 20°C $\xrightarrow[\Delta S_1]{}$ 1 kg Water at 0°C $\xrightarrow[\Delta S_2]{}$ 1 kg Ice at 0°C $\xrightarrow[\Delta S_3]{}$ 1 kg ice at −10°C

$$\boxed{\Delta S_1 = mC_p \log_e\left(\frac{T_2}{T_1}\right)}$$

given $m = 1\ kg$

$C_p = (\text{of Water}) = 4200\ Jkg^{-1}K^{-1}$

$T_1 = 273 + 20 = 293\ K$ & $T_2 = 273\ K$

$\Delta S_1 = 1\ kg \times 4200\ Jkg^{-1}K^{-1} \log_e \frac{273}{293}$

$$\boxed{\Delta S_1 = -2.9694 \times 10^2\ JK^{-1}}$$

$$\boxed{\Delta S_2 = -\frac{mL}{T}}$$ (−ve sign as entropy will reduce)

given $m = 1\ kg$

$L = 335 \times 10^3\ J\,kg^{-1}$, $T = 273\ K$

$$\boxed{\Delta S_2 = -1.2271 \times 10^3\ JK^{-1}}$$

$$\Delta S_3 = m C_p \log_e \frac{T_2}{T_1}$$

C_p (for ice) $= 2100\ J\,kg^{-1}K^{-1}$

$T_2 = 263\,K,\ T_1 = 273\,K$

$$\boxed{\Delta S_3 = -7.8367 \times 10^{1}\ JK^{-1}}$$

Therefore, The Total Change in Entropy

$$\Delta S_t = \Delta S_1 + \Delta S_2 + \Delta S_3$$

$$\boxed{\Delta S = -1.6024 \times 10^{3}\ JK^{-1}}$$

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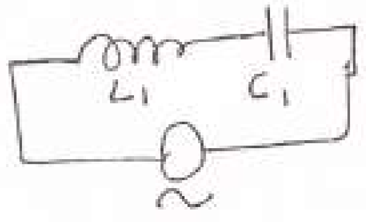
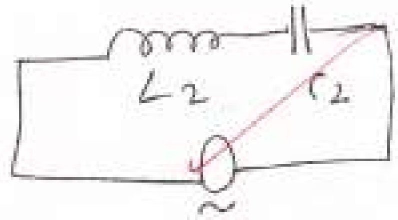
The fall in entropy is expected as solid state (Ice) is less disordered compared to liquid state (water).

c) The resonant frequency of an LC circuit is

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$





Resonant frequency, $\omega_1 = \frac{1}{\sqrt{L_1 C_1}}$

Resonant frequency, $\omega_2 = \frac{1}{\sqrt{L_2 C_2}}$

Given, the two circuits have the same resonant frequency

Therefore, $\omega_1 = \omega_2 \Rightarrow \boxed{L_1 C_1 = L_2 C_2}$

When all the elements are connected in series:-

Resonant frequency, $\omega' = \dfrac{1}{\sqrt{L_T C_T}}$

L_T (series of combination of L_1 & L_2) $= L_1 + L_2$

C_T (series combination of C_1 & C_2) $\Rightarrow \dfrac{1}{C_T} = \dfrac{1}{C_1} + \dfrac{1}{C_2} \Rightarrow C_T = \dfrac{C_1 C_2}{C_1 + C_2}$

$$\omega' = \frac{1}{\sqrt{(L_1+L_2)\left(\frac{C_1 C_2}{C_1+C_2}\right)}} = \frac{1}{\sqrt{\frac{L_1C_1C_2 + L_2C_1C_2}{C_1+C_2}}}$$

from ① $L_1C_1 = L_2C_2$

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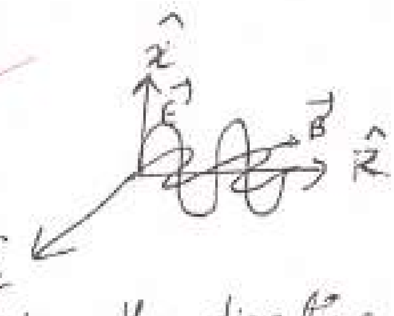
$$\omega' = \frac{1}{\sqrt{\frac{L_2C_2C_2 + L_2C_1C_2}{C_1+C_2}}} = \frac{1}{\sqrt{\frac{L_2C_2(C_2+C_1)}{C_1+C_2}}}$$

$$\boxed{\omega' = \frac{1}{\sqrt{L_2C_2}} = \omega_1 = \omega_2}$$

Hence, all circuits have same Resonant frequency.

d) Let the Electric field and Magnetic field in the dielectric medium be as follows -

$$\vec{E} = E_0 \cos(kz - \omega t)\,\hat{x}$$
$$\vec{B} = B_0 \cos(kz - \omega t)\,\hat{y}$$



(The wave is propagating along the z-axis, the directions are appropriately choosen as $\vec{E}$ & $\vec{B}$ are perpendicular to each other & wave due to transverse nature)

The Poynting Vector is defined as the energy flow per unit time per unit area.

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu}$$

$$\vec{S} = \frac{\left[E_0 \cos(kz - \omega t)\hat{x}\right] \times B_0 \cos(kz-\omega t)\hat{y}}{\mu}$$

$$\vec{S} = \frac{E_0 B_0}{\mu} \cos^2(kz - \omega t)\underbrace{(\hat{x} \times \hat{y})}_{\hat{z}}$$

Now $\vec{B} = \dfrac{\hat{z} \times \vec{E}}{v}$ where $v = \dfrac{1}{\sqrt{\varepsilon\mu}} = \dfrac{\omega}{K}$

$\varepsilon \rightarrow$ electric Permeability

$\mu \rightarrow$ Magnetic Permeability

Therefore, $B_0 = \dfrac{KE_0}{\omega}$

$$\boxed{\vec{S} = \hat{z}\,\frac{kE_0^2}{\omega\mu}\cos^2(kz - \omega t)}$$

Therefore, energy is transmitted in the direction of wave propagation

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Q8 (a) According to the law of equipartition of energy, the average energy associated with each degree of freedom is $\frac{K_B T}{2}$.

Let a system a free gas particles has f degrees of freedom. Then, Internal energy for one mole of gas particles,

$$U = f \times N_A \left(\frac{K_B T}{2}\right) = \frac{f N_A K_B T}{2} = \frac{f R T}{2} \qquad \left[K_B = \frac{R}{N_A}\right]$$

C_V = Specific Heat Capacity at Constant Volume

$$C_V = \left(\frac{\partial U}{\partial T}\right) \Rightarrow \boxed{C_V = \frac{f R}{2}} \quad -①$$

C_P = Specific Heat Capacity at Constant Pressure

According to Mayer's formula, for one mole,

$$\boxed{C_P = C_V + R = \frac{f R}{2} + R = \left(\frac{f}{2} + 1\right) R} \quad -②$$

6/10

from ① & ②, $\frac{C_P}{C_V} = \frac{\frac{f}{2} + 1}{\frac{f}{2}} = \frac{f+2}{f} = 1 + \frac{2}{f}$

$$\boxed{f = \frac{2}{\frac{C_P}{C_V} - 1}}$$

$\frac{C_P}{C_V} = \gamma$. Therefore, we can relate γ & f as $\boxed{f = \frac{2}{\gamma - 1}}$

b). Fermi-Dirac Distribution | Bose-Einstein Distribution

Fermi-Dirac Distribution	Bose-Einstein Distribution
→ Obeyed by fermions.	→ Obeyed by Bosons.
→ Half-Integral Spin Particles	→ Integral Spin Particles
→ Obey Pauli's Exclusion Principle	→ Do not obey Pauli's Exclusion Principle
→ Have Anti Symmetric Wave function.	→ Have Symmetric Wave function

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The Distribution function,

$$\boxed{f(E) = \frac{1}{\exp\left(\frac{E-\mu}{k_BT}\right) + K}}$$

where $K = 0$ for Maxwell-Boltzmann Distribution
$= +1$ for Fermi-Dirac Distribution
$= -1$ for Bose-Einstein Distribution

If $f(E) << 1$, then $\exp\left(\frac{E-\mu}{k_BT}\right) + K >> 1$

Hence, $\exp\left(\frac{E-\mu}{k_BT}\right) + K \simeq \exp\left(\frac{E-\mu}{k_BT}\right)$

$$\boxed{f(E) = \exp\left(\frac{\mu-E}{k_BT}\right)}$$

This is observed when fermions & bosons lose their characteristic feature of Indistinguishability.

(c) Thermodynamic systems have various properties like entropy (S), Internal energy (U), enthalpy (H), etc. which can not be measured directly.

Maxwell's Relations relate this properties with directly measurable quantities, for eg., Pressure (P), Temperature (T), Volume (V)

The four Maxwell's Relations are :-

1) $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$

2) $\left(\frac{\partial V}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$

~~why need to write~~

3) $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$

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4) $\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S$

Clausius- Clapeyron equation gives up the change in temperature with the change in pressure.

Let two phases be in stable equilibrium. If small heat δQ is provided then small mass dm will undergo change in phase.

$\delta Q = L\,dm$ (L – Latent heat)

$\partial V = (V_2 - V_1)dm$ (V_2 & V_1 is specific volume in the two phases.

from the first Maxwell's Relations,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

$$T\left(\frac{\partial S}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V$$

$$\left(\frac{\delta Q}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V \qquad \left[\partial S = \frac{\delta Q}{T}\right]$$

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from above, $\frac{L\,dm}{(V_2 - V_1)\,dm} = T\left(\frac{dp}{dT}\right)$

$$\boxed{\frac{dp}{dT} = \frac{L}{T(V_2 - V_1)}}$$

This is the required Clausius-Clapeyron equation which is followed by all first order phase transitions to a very high degree. It explains like Better Cooking in Pressure Cooker, ~~or~~ Difficulty in Cooking in higher altitudes due to low pressure, etc

Other Applications of Maxwell's Relation

→ Gives $\frac{\partial C_p}{\partial p}$ & $\frac{\partial C_v}{\partial V}$

→ Gives the TdS Equations

→ Gives Energy Equations, i.e, $\left(\frac{\partial U}{\partial V}\right)_T$, $\left(\frac{\partial U}{\partial p}\right)_T$

→ Explains why $C_p - C_v \to 0$ as $T \to 0$

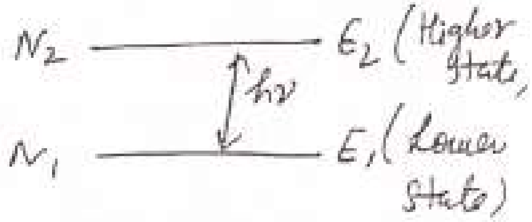
~~→ Explains why~~

d) According to the Maxwell-Boltzmann Distribution, the number of particles N in energy level E is directly proportional to $\exp\left(-\frac{E}{k_BT}\right)$.

Therefore, $\boxed{N = N_0 \exp\left(\frac{-E}{k_BT}\right)}$ where N_0 is constant

Let there be two energy levels E_1 & E_2 with N_1 & N_2 particles such that $\boxed{E_2 - E_1 = h\nu}$

$$\frac{N_2}{N_1} = \exp\left(\frac{-(E_2-E_1)}{k_BT}\right)$$



$$\frac{N_2}{N_1} = \exp\left(\frac{-\Delta E}{k_BT}\right)$$

At equilibrium, the Number of Particles in Lower energy state N_1 is always greater than those in higher energy state.

$$\frac{N_2}{N_1} < 1 \Rightarrow \exp\left(\frac{-\Delta E}{k_BT}\right) < 1$$

~~This gives~~ $\boxed{\frac{\Delta E}{k_BT} > 0}$

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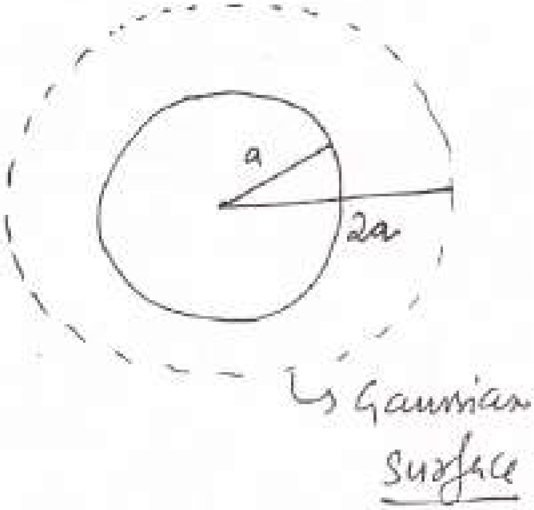
This gives that T is always Positive, ie, There cannot be any Negative Absolute Temperature.

Such a condition can only be realised by Population Inversion, ie, $N_2 > N_1$. This is achieved by utilising Metastable states in Lasers.

Q1(a) According to the gauss's law of electrostatics, the flux of electric field around a surface S is equal to $(1/\varepsilon_0)$ times the charge enclosed by the surface.

$$\boxed{\int_S \vec{E}\cdot\vec{da} = \frac{q_{enc}}{\varepsilon_0}} = -4\pi G(M)$$



$$q_{encl} = \int \rho\, dv = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} \frac{\rho_0 a}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$q_{enclos} = \rho_0 a\, 4\pi \int_0^a r\,dr$$

$$\boxed{q_{enc} = \frac{\rho_0 4\pi a^3}{2}} \text{ — (1)}$$

$$\int_S \vec{E}\cdot\vec{da} = \int_S E\hat{r}\cdot da\,\hat{r} = E\int da = E\,4\pi(2a)^2$$

$$\boxed{\int_S \vec{E}\cdot\vec{da} = E4\pi(4a^2)} \text{ — (2)}$$

from (1) & (2), $E(4\pi)(4a^2) = \frac{\rho_0 4\pi a^3}{2}$

$$\boxed{\vec{E} = \frac{\rho_0 a}{8}\hat{r}}$$

Gravitational field has been asked?

This is the required electric field.

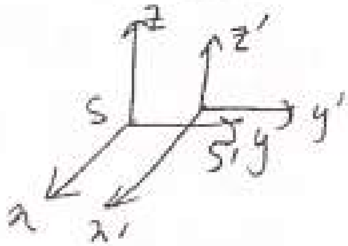
0

b) According to the Lorentz Transformation for Velocity Addition

$$\boxed{u_x' = \frac{u_x - V}{1 - \frac{u_x V}{c^2}}}$$

Let S & S' be two Inertial frame of reference with parallel co-ordinate axes & S' moving with speed v w.r.t. S.

Let, the Speed of Object in S-frame be c.



Then, Speed of Object in S'-frame

$$u_x' = \frac{c - V}{1 - \frac{c \times V}{c^2}} = \frac{c(c-V)}{c-V}$$

$$\boxed{u_x' = c}$$

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Therefore, even after addition of, the velocity of light is reproduced. This is in line with the Postulate of Special Theory of Relativity, that Speed of Light is constant in all Inertial frames of reference.

$$u_x' = \frac{c + V}{1 + \frac{V}{c}} = c$$

$$u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}} =$$

(c) The Angular Velocity of the Earth

$$\omega = \frac{2\pi}{24\,hours} = \frac{2\pi}{24 \times 60 \times 60\,sec}$$

$$\boxed{\omega = 7.272 \times 10^{-5}\ rad\ s^{-1}}$$

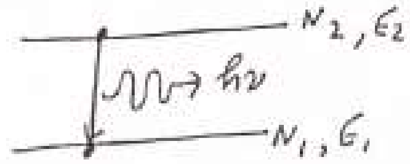
1

This can be calculated by using the phenomenon of aberration of stars which is relativistic in nature. According to it, the trajectory of stars overhead, ie, North Pole is circular compared to other stars which is elliptical.

Focault's pendulum

$$T = \frac{2\pi}{\omega \sin\lambda}$$

d) Einstein's A-Coefficient gives the Probability of Spontaneous Emission of Photon as particle de-excites from Higher Energy State to Lower energy-State.



Rate of Spontaneous Emission, $\boxed{R_{sp} = A_{21} N_2}$

The A-Coefficient physically signifies the Lifetime of the excited energy state. It is inversely proportional to the Lifetime.

$$\boxed{A \propto \frac{1}{t_{exci}}}$$

Therefore, Metastable States have low value for A-coefficient & hence, Rate of Spontaneous emission. This is why they are used in Lasers to achieve Population Inversion.

Rate of Stimulated Emission, $R_{stim} = B_{21} N_2 \rho(\nu)$
↳ Photon Density

$6\frac{1}{2}$

10

$$\boxed{\frac{R_{stim}}{R_{sp}} = \frac{B_{21}\,\rho(\nu)}{A_{21}}}$$

we know $A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21}$ & $\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}$

$$\frac{R_{stim}}{R_{sp}} = \frac{1}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}$$

Therefore, Higher the frequency, lower the Rate of ~~Spontaneous~~ Stimulated Emission w.r.t. Spontaneous. Hence, Lasing Action is difficult for X-ray (high ν) compared to Infrared (Low ν).

(c) Pulse Dispersion can be defined as time variation between the pulses which are travelling in different order modes.

Higher Order Modes → Greater distance → ~~Lesser~~ More time

Lower Order Modes → Lesser Distance → Less time

Pulse Dispersion, $\Delta t = \frac{n_1 L}{c}\left(\frac{n_1}{n_2} - 1\right)$

where n_1 = Refractive Index of Core = 1.5

n_2 = Refractive Index of Cladding

fractional Index Difference $\Delta = \frac{n_1 - n_2}{n_1} = 0.001$

$\boxed{n_2 = 1.4985}$

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$L = 1\ km$

Pulse Dispersion ⇒ $\boxed{\Delta t = 5.005\ ns}$

for ~~Tx~~ $L = 2\ km$ ⇒ $\boxed{\Delta t = 10\ ns}$

Therefore, the minimum pulse ~~dis~~ separation that can be transmitted is 10 ns. If reduced further due to pulse dispersion, signal will not be intercepted correctly at the receiver end & there will be distortion.