DIAS Delhi Institute for Administrative Services
India's Leading Institute for Civil Services Examination

CIVIL SERVICES EXAMINATION (MAINS) 2020
DIAS PHYSICS OPTIONAL TEST SERIES
Test X : FULL SYLLABUS TEST PAPER - II

Time Allowed: 3 Hours **Max Marks: 250**

Attempt five question in all, choosing at least one question per section and question number 1 and 5 being compulsory.

Q1. (a) Find out the probability current density for a particle in state $\psi(x,t) = 2e^{i(kx-\omega t)}$ and explain the answer.

(b) In which part of spectrum ESR and NMR lie? Find out Lande's factor for an electron if it is known that fixed field of 9.5×10^{7} Tesla produces ESR at frequency 340 KHz.

$\Delta E = \mu_B B g \qquad g = \dfrac{h\gamma}{\mu_B B}$

(c) Show that for Linear Harmonic Oscillator zero point energy is in accord with Heisenberg's uncertainty principle wave function in ground state is $\psi(0) = \left(\dfrac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$. Find its value in state 1.

$$\psi_n = xe^{-\frac{m\omega}{2\hbar}x^2} a + |n\rangle = \sqrt{n+1}|n+1\rangle$$

(d) (i) Write down the commutation results satisfied by angular momentum operators. Find out the values of $[L_z, x]$ and $[L_z, y]$.

(ii) Show that $e^{i\theta\vec{\sigma}\cdot\hat{n}} = \cos\theta + i\vec{\sigma}\cdot\hat{n}\sin\theta$, where σ is Pauli spin operator.

(e) Derive an expression for frequency of the electron orbiting in ground state of Hydrogen atom from Bohr's model.

(10 × 5 = 50 Marks)

Q2. (a) Determine the energy levels and the corresponding normalized eigen functions for a particle in 1 D potential of the form:

$$V(x) = \infty \qquad a < x < 0$$
$$V(x) = 0 \qquad 0 < x < a$$

Is the wave function continuous everywhere? Show for n th states

$$\Delta x = \frac{a^2}{12}\left[1 - \frac{6}{n^2\pi^2}\right].$$

(b) Find out the total energy of Fermions at $0^0 K$ in a cubical box, where all terms have their usual meaning. (30 + 20 = 50 Marks)

Q3. (a) Set up and solve angular momentum problem $L^2 Y(\theta,\phi) = \lambda\hbar^2 Y(\theta,\phi)$

(b) What is Born-Openheimer approximation ? Under what conditions does it break down ? Discuss the origin of O, P, Q, R, S branches in vibration-rotation spectra? down?

(25 + 25 = 50 Marks)

Q4. (a) Find out an expression for precessional frequency of a nucleus? Why are two fields necessary? What is chemical shift and how can it be studied? Find the NMR frequency for ^{13}C nuclei. (g = 1.404) at field strength of 2.34 Tesla. (7 + 3 + 5 + 5)

(b) Wave number in a band system are given by $\bar{\omega} = 24762 + 25m - 2.1m^2\,cm^{-1}$. Deduce the position of band head and the values of B' and B'' and comment on the results.

(c) How many revolutions per second does a $C^{12}O^{16}$ molecule make when J quantum numbers are 1 and 10 respectively. Given inter-nuclear separation is $1.128\overset{0}{A}$. What is the separation between the lines observed in Microwave spectra? (20 + 15 + 15 = 30 Marks)

SECTION - B

Q5. (a) Define Q of a reaction. Calculate the Q-value of the reaction:

$^9_4Be\left(^4_2He, n\right)^{12}_6C$

Given: Mass $\left(^9Be\right) = 9.012183\ u$

Mass $\left(^4He\right) = 4.002603\ u$

Mass $\left(^{12}C\right) = 12.000\ u$

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(b) Show that any arbitrary rotation axis is not permitted in a crystal lattice.

(c) State the quantum numbers I_3, Y and S for uds quarks and antiquarks. Which combination of these leads to the formation of a (i) proton and (ii) neutron ?

(d) Simplify the logical expression $\left[A\bar{B}(C+BD)+\bar{A}\bar{B}\right]C$.

(e) Differentiate between n-p-n and p-n-p transistors. Give their device structure and biasing circuits when used as an amplifier. (10 × 5 = 50 Marks)

Q6. (a) Describe grand unification theories (GUT).

(b) How many types of neutrinos exist ? How do they differ in their masses ?

(c) What are type I and type II superconductors? A type I superconductor with $T_c = 7K$ has slope $\frac{dB_c}{dT} = -25\,mT/K$ at transition temperature T_c. Estimate its critical field at $6K$.

(d) Calculate the packing fraction for $_4Be^7$ nucleus. Given $a_1 = 14.1\,MeV$, $a_2 = 13\,MeV$, $a_3 = 0.6\,MeV$, $a_4 = 19\,MeV$ and $1\,amu = 931\,MeV$. (10 + 10 + 15 + 15 = 50 Marks)

Q7.(a) Which selection rules allow γ- transition to occur? How do γ- radiations interact with matter and how their energy can be measured?

(b) Describe an operational amplifier based integrator. Using operational amplifier integrators, design a circuit to solve the following differential equation:

$$\frac{d^2V}{dt^2} + 2\frac{dV}{dt} + 3V = 0$$

(c) Draw the device structure of a p-n junction solar cell and explain how light energy is converted into electrical energy. Draw and explain its I-V characteristics.

(15 + 20 +15 = 50 Marks)

Q8. (a) Find an expression for lattice specific heat of a solid, and its low and high temperature limits. What is Debye temperature ?

(b) Describe the motion of an electron in one dimensional periodic potential and show that it leads to formation of bands of allowed and forbidden states in the electron energy spectrum. How are the conductors, semiconductors and insulators discriminated on the basis of band structure?

(c) Distinguish between a superconductor and perfect conductor. Explain what is a Cooper pair.

(20 + 20 + 10 = 50 Marks)

22/12/2020

Name - Mamta Yadav
Roll No. - 0829179
Paper - II Full Length Test

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Q 1	Q 2	Q 4	Q 5	Q 8
23 ½	31	27	30 ½	32

144/250

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Q2. The given potential is that of an Infinite Potential Well.

$$V(x) = \begin{cases} \infty, & x < 0,\ x > a \\ 0, & 0 < x < a \end{cases}$$

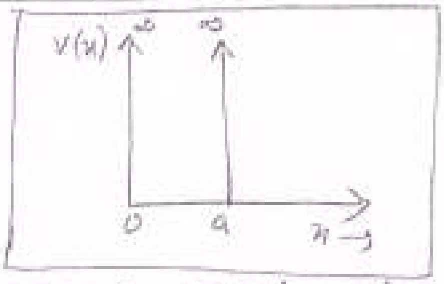

Fig. Infinite Potential Well

The time independent Schroedinger Equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

In region $a < x < 0$, $V(x) = \infty$. Therefore, $\boxed{\psi_{out}(x) = 0}$ — (1)

In region $0 < x < a$, $V(x) = 0$,

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$$\frac{d^2\psi(x)}{dx^2} + K^2\psi(x) = 0 \quad \text{where } k^2 = \frac{2mE}{\hbar^2}$$

$$\boxed{\psi_{in}(x) = A\sin kx + B\cos kx} \text{ — (2)}$$

According to the Born Interpretation of wave function, the wave function is continuous everywhere. Therefore,

$\psi_{in}(x) = \psi_{out}(x)$ at $x = 0$ $\Rightarrow$ $\boxed{0 = B}$ — (3)

$\psi_{in}(x) = \psi_{out}(x)$ at $x = a$ $\Rightarrow$ $A\sin ka = 0$

$\sin ka = 0 = \sin n\pi$ (for non-trivial Solution of Schrodinger Equation)

$$\boxed{K_n = \frac{n\pi}{a}} \text{ — (4)}$$

~~using (1), (2), (3), (4)~~ A can be obtained from the Normalisation Condition according to which $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$

$$\int_0^a |A|^2 \sin^2\frac{n\pi x}{a}\,dx = 1$$

$$\frac{|A|^2}{2}\int_0^a \left(1 - \cos\frac{2n\pi x}{a}\right)dx = 1 \Rightarrow \frac{|A|^2}{2}\left[x - \frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right]_0^a = 1$$

$$\langle x\rangle = \frac{1}{a}\left[\frac{a^2}{2} + \frac{a}{2n\pi}\left(\frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right)\Bigg|_0^a\right]$$

$= 0$ [as $\cos 0 = \cos 2n\pi = 1$]

$$\boxed{\langle x\rangle = \frac{a}{2}}$$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty}\psi^*(x)\,x^2\,\psi(x)\,dx = \int_0^a \frac{2x^2}{a}\sin^2\frac{n\pi x}{a}\,dx$$

$$\langle x^2\rangle = \frac{1}{a}\int_0^a x^2\left(1-\cos\frac{2n\pi x}{a}\right)dx = \frac{1}{a}\left[\int_0^a x^2\,dx - \int_0^a x^2\cos\frac{2n\pi x}{a}\,dx\right]$$

$$\langle x^2\rangle = \frac{1}{a}\left[\frac{x^3}{3}\Bigg|_0^a - \left(x^2\frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a - \int_0^a \frac{2x\sin\frac{2n\pi x}{a}\,dx}{\frac{2n\pi}{a}}\right)\right]$$

$$\langle x^2\rangle = \frac{1}{a}\left[\frac{a^3}{3} + \frac{a}{n\pi}\left(\frac{-x\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a - \int_0^a \frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right)\right]$$

$$\langle x^2\rangle = \frac{1}{a}\left[\frac{a^3}{3} + \frac{a}{n\pi}\left(\frac{-a^2}{2n\pi} + \frac{a}{2n\pi}\,\frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a\right)\right] = \frac{1}{a}\left[\frac{a^3}{3} - \frac{a^3}{2n^2\pi^2}\right]$$

$$\boxed{\langle x^2\rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}}$$

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$$\Delta x = \sqrt{\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}} = \sqrt{\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}}$$

$$\boxed{\Delta x = \sqrt{\frac{a^2}{12}\left(1 - \frac{6}{n^2\pi^2}\right)}}$$

This is the required answer.

$$E = \frac{8\pi V}{5}\left(\frac{2m}{h^2}\right)^{3/2} E_F^{5/2}$$

This is the required expression for total energy.

Further, the Total Number of Fermions at $0^\circ K$ is

$$N = \int_0^\infty dn = \int_0^\infty D(E)f(E)dE = \int_0^{E_F} 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2} dE$$

$$N = \frac{8\pi V}{3}\left(\frac{2mE_F}{h^2}\right)^{3/2}$$

using this above, $E = \frac{3}{5} N E_F$

The energy of the Fermion System is quite high at $0^\circ K$ because fermions obey pauli's exclusion principle. According to it, no two fermions can occupy the same energy state. Hence, even at 0K sufficiently high Number of energy states are occupied.

$12\frac{1}{2}$ / 20

22/12/2020

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Paper - II Full Length Test

Q1	Q2	Q4	Q5	Q8
23½	31	27	30½	32

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DK

Q2. The <u>given potential</u> is that of an <u>Infinite Potential Well</u>.

$$V(x) = \begin{cases} \infty, & x<0,\ x>a \\ 0, & 0<x<a \end{cases}$$

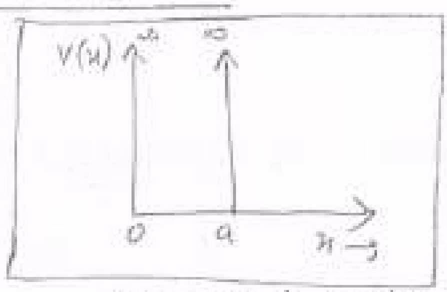

Fig. Infinite Potential Well

The <u>time independent Schroedinger Equation</u> is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

In region $a<x<0$, $V(x) = \infty$. Therefore, $\boxed{\psi_{out}(x) = 0}$ — (1)

In region $0<x<a$, $V(x) = 0$.

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$$\frac{d^2\psi(x)}{dx^2} + K^2\psi(x) = 0 \quad \text{where } K^2 = \frac{2mE}{\hbar^2}$$

$$\boxed{\psi_{in}(x) = A\sin kx + B\cos kx} \text{ — (2)}$$

According to the <u>Born Interpretation</u> of <u>wave function</u>, the <u>wave function</u> is <u>continuous</u> everywhere. Therefore,

$\psi_{in}(x) = \psi_{out}(x)$ at $x=0$ $\Rightarrow$ $\boxed{0 = B}$ — (3)

$\psi_{in}(x) = \psi_{out}(x)$ at $x=a$ $\Rightarrow$ $A\sin ka = 0$

$\sin ka = 0 = \sin n\pi$ (for <u>non-trivial Solution</u> of <u>Schrodinger Eqn</u>)

$$\boxed{K_n = \frac{n\pi}{a}} \text{ — (4)}$$

~~using (1), (2), (3), (4)~~ <u>A</u> can be obtained from the <u>Normalisation Condition</u> according to <u>which</u>

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

$$\int_0^a |A|^2 \sin^2\frac{n\pi x}{a} dx = 1$$

$$\frac{|A|^2}{2}\int_0^a \left(1 - \cos\frac{2n\pi x}{a}\right) dx = 1 \Rightarrow \frac{|A|^2}{2}\left[x - \frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right]_0^a = 1$$

$$A = \sqrt{\frac{2}{a}}$$

Therefore, the Normalised eigen function for the Particle is

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}, & 0 < x < a \\ 0, & a < x < 0 \end{cases}$$

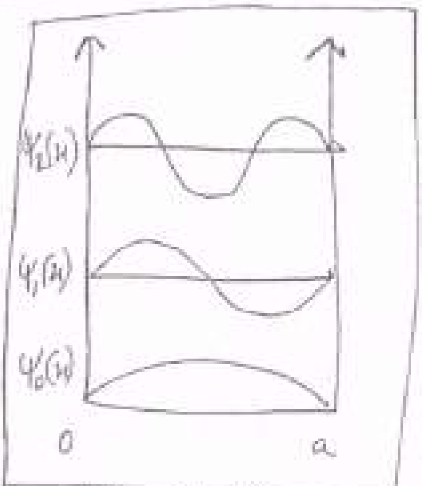

Fig. First few Wave functions

The Energy Eigen-Value, $E_n = \frac{\hbar^2 k^2}{2m}$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Therefore, the energy is → Discrete
→ Infinite no. of energy states
→ Non-zero in ground state in accordance with heisenberg's uncertainity principle

The Wave Function is continuous everywhere including the boundaries, ie, $x = 0$ and a as required by Born Interpretation.

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\langle x\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, x\, \psi(x)\, dx = \int_0^a \frac{2x}{a} \sin^2\frac{n\pi x}{a}\, dx$$

[The Wave function is zero for other values of x]

$$\langle x\rangle = \frac{2}{2} \cdot \frac{1}{a}\int_0^a x\left(1 - \cos\frac{2n\pi x}{a}\right) dx$$

$$\langle x\rangle = \frac{1}{a}\left[\int_0^a x\,dx - \int_0^a x\cos\frac{2n\pi x}{a}\,dx\right] = \frac{1}{a}\left[\frac{x^2}{2}\Big|_0^a - \left(\frac{x \sin\frac{2n\pi x}{a}}{\frac{2n\pi x}{a}}\Big|_0^a - \int_0^a \frac{\sin\frac{2n\pi x}{a}\,dx}{\frac{2n\pi x}{a}}\right)\right]$$

as $\sin 2n\pi a = \sin 0 = 0$

$$\langle x \rangle = \frac{1}{a}\left[\frac{a^2}{2} + \frac{a}{2n\pi}\left(\frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right)\Bigg|_0^a\right]$$

$$\frac{2n\pi}{a} \to 0 \quad [\text{as } \cos 0 = \cos 2n\pi = 1]$$

$$\boxed{\langle x \rangle = \frac{a}{2}}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, x^2\, \psi(x)\, dx = \int_0^a \frac{2x^2}{a}\sin^2\frac{n\pi x}{a}\, dx$$

$$\langle x^2 \rangle = \frac{1}{a}\int_0^a x^2\left(1 - \cos\frac{2n\pi x}{a}\right)dx = \frac{1}{a}\left[\int_0^a x^2\,dx - \int_0^a x^2\cos\frac{2n\pi x}{a}\,dx\right]$$

$$\langle x^2 \rangle = \frac{1}{a}\left[\frac{x^3}{3}\Bigg|_0^a - \left(\frac{x^2 \sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a - \int_0^a \frac{2x\sin\frac{2n\pi x}{a}\,dx}{\frac{2n\pi}{a}}\right)\right]$$

$$\langle x^2 \rangle = \frac{1}{a}\left[\frac{a^3}{3} + \frac{a}{n\pi}\left(\frac{-x\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a - \int_0^a \frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right)\right]$$

$$\langle x^2 \rangle = \frac{1}{a}\left[\frac{a^3}{3} + \frac{a}{n\pi}\left(\frac{-a^2}{2n\pi} + \frac{a}{2n\pi}\,\frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\Bigg|_0^a\right)\right] = \frac{1}{a}\left[\frac{a^3}{3} - \frac{a^3}{2n^2\pi^2}\right]$$

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$$\boxed{\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}}$$

$$\Delta x = \sqrt{\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}} = \sqrt{\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}}$$

$$\boxed{\Delta x = \sqrt{\frac{a^2}{12}\left(1 - \frac{6}{n^2\pi^2}\right)}}$$

This is the required answer.

b) Fermions are half-integral spin particles which have the following properties → They obey Fermi-Dirac Distribution.

→ They follow Pauli's Exclusion Principle.

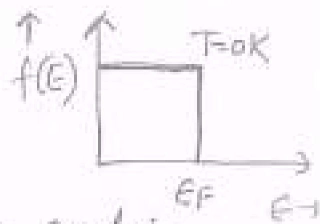
→ They have anti-symmetric wave function.

The Occupancy function for Fermions according to the Fermi-Dirac Distribution

$$f(E) = \frac{1}{\exp\left(\frac{E-E_F}{k_BT}\right)+1}$$

E_F is the Fermi-energy, i.e. the highest occupied energy level at T=0K.

Therefore, at T=0K $\quad f(E) = \begin{cases} 0, & \text{if } E > E_F \\ 1, & \text{if } E < E_F \end{cases}$



The Density of States $D(E)$, ie, the Number of Particles occupying energy in the interval E to $E+dE$ per unit energy interval dE.

$$D(E) = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}$$

The Total Energy of Fermions at 0°K in a Cubical Box is

$$E = \int_0^\infty E D(E) f(E)\, dE = \int_0^{E_F} E D(E) \underset{=1}{F(E)}\, dE + \int_{E_F}^{\infty} E D(E) \underset{=0}{F(E)}\, dE$$

$$E = \int_0^{E_F} 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{3/2}\, dE$$

So $E = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} \frac{E_F^{5/2}}{5/2}$

$$E = \frac{8\pi V}{5}\left(\frac{2m}{h^2}\right)^{3/2} E_F^{5/2}$$

This is the required expression for total energy.

Further, the Total Number of Fermions at 0K is

$$N = \int_0^{\infty} dn = \int_0^{\infty} D(E)f(E)\,dE = \int_0^{E_F} 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}\,dE$$

$$N = \frac{8\pi V}{3}\left(\frac{2mE_F}{h^2}\right)^{3/2}$$

using this above, $E = \frac{3}{5} N E_F$

The energy of the Fermion System is quite high at 0K because fermions obey pauli's exclusion principle. According to it, no two fermions can occupy the same energy state. Hence, even at 0K sufficiently high Number of energy states are occupied.

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Q4. (a) The Nucleus has angular momentum Vector ($\vec{I}$) when this nucleus is placed in external magnetic field, the Magnetic moment of nucleus interacts with the Magnetic field and a torque acts. This torque is perpendicular to $\vec{I}$ and hence causes it a precess a cone with $\vec{B}$ as axis. This is known as Precessional Motion and the corresponding frequency is Precessional frequency of a Nucleus.

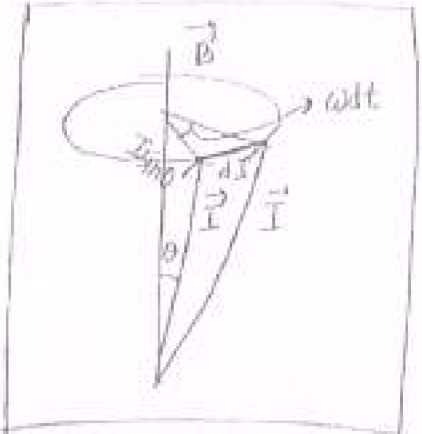


Fig. Precession of $\vec{I}$ about $\vec{B}$

We know, $\vec{\tau} = \vec{\mu}_I \times \vec{B} = \mu_I B \sin\theta$ —— (1)

From the figure, $dI = (\omega dt)(I \sin\theta)$

$$\frac{dI}{dt} = \omega I \sin\theta$$

$$\tau = \frac{dI}{dt} = \omega I \sin\theta \quad \text{—— (2)}$$

from (1) & (2), $\omega I \sin\theta = \mu_I B \sin\theta$

$$\boxed{\omega = \left(\frac{\mu_I}{I}\right) B}$$

$\frac{\mu_I}{I} = g_N \left(\frac{e}{2m_p}\right)$ where $g_N \to$ g-factor of the nucleus

Therefore, the Precessional Frequency of the Nucleus $\boxed{\omega = g_N \left(\frac{eB}{2m_p}\right) \text{ or } \nu = g_N \left(\frac{eB}{4\pi m_p}\right)}$

Why are Two fields Necessary?

In Nuclear Magnetic Resonance, two fields are needed. The $\vec{I}$ can take only certain discrete orientations w.r.t. the $\vec{B}$.

This is known as Space Quantisation. Hence, due to ~~another~~ Magnetic Field, the Nuclear Levels are split into $(2I+1)$ for each M_I from $-I, \ldots, +I$.

When another magnetic field is applied whose frequency is equal to Precessional frequency of Nucleus, the nucleus flips from one state to another. This is called NMR. Hence, two fields are needed - one for splitting energy levels & another for Resonance.

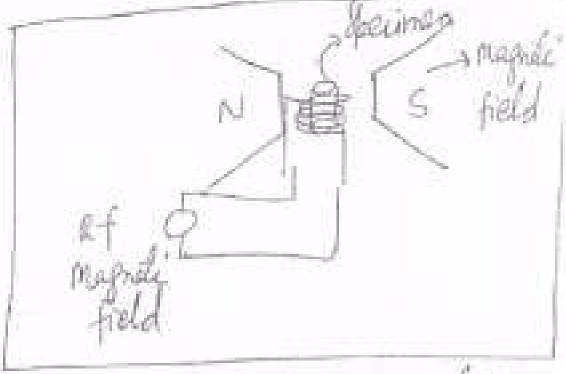


Fig. Nuclear Magnetic Resonance

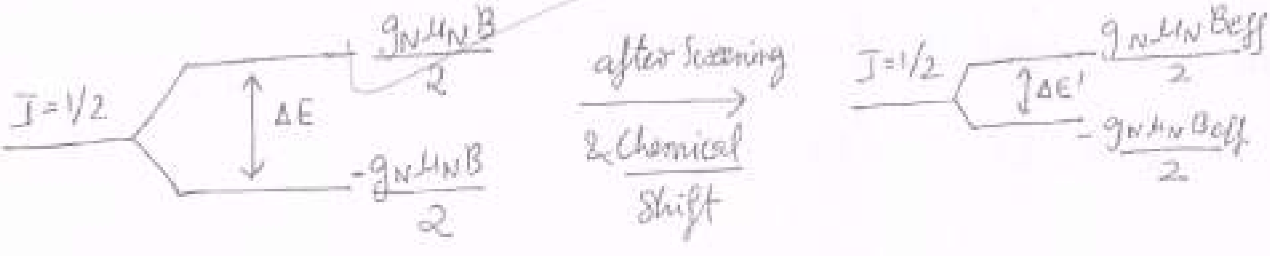
CHEMICAL SHIFT

Nuclear Magnetic Resonance is used for probing molecular structure &

The Precessional frequency of certain standard atoms/molecules is known. Hence NMR frequency changes due to the effect of Local magnetic field and the Screening effects of electrons & other atoms. This shifts the energy levels and hence NMR condition. This is known as Chemical Shift.

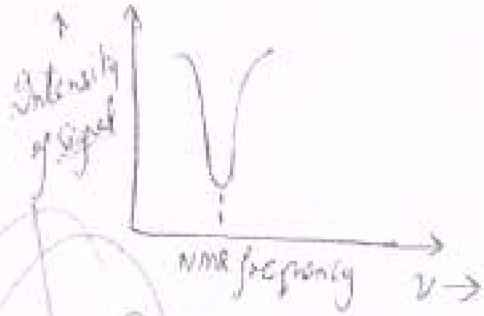
$$\omega = g_N\left(\frac{eB_{eff}}{2m_p}\right) = g_N\left(\frac{eB(1-\sigma)}{2m_p}\right)$$

where σ = Screening Coefficient causing chemical shift.



Numerical, ~~Precessional~~ NMR frequency = Precessional frequency

$$\boxed{\nu = g_N \left(\frac{eB}{4\pi m_p}\right)}$$



Given: $g_N = 1.404$

$B = 2.34\,T$

$m_p = 1.67 \times 10^{-27}$ kg

$$\boxed{\nu = 25.0482 \text{ MHz}}$$ This is the Resonance frequency

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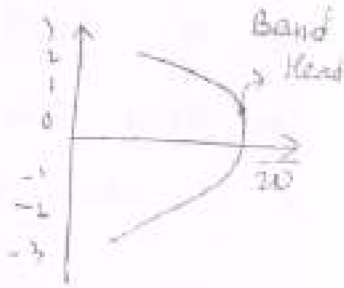
b. The Master Equation of Wave-Number for Rotational-Vibrational Spectra is -

$$\boxed{\bar{\omega} = \omega_0 + (B_{v'} - B_{v''})m^2 + (B_{v'} + B_{v''})m}$$

Given, $\bar{\omega} = 24762 + 25m - 2.1m^2$ cm^{-1}



For Band Head, $\frac{d\bar{\omega}}{dm} = 0$

$0 + 25 - 4.2m = 0$

$$\boxed{\begin{aligned} m &= 5.9524 \\ m &\simeq 6 \end{aligned}}$$

Therefore, the Band Head corresponds to $m = 6$, ie,

$\bar{\omega} = 24762 + 25(6) - 2.1(6)^2$

$$\boxed{\bar{\omega}_{\text{Band-Head}} = 24836.4 \text{ cm}^{-1}}$$

for B' and B'', on comparing with the [illegible] equation

$$B_{v'} + B_{v''} = 25\ cm^{-1}$$

$$B_{v'} - B_{v''} = -2.1\ cm^{-1}$$

On solving, $B_{v'} = 11.45\ cm^{-1}$, $B_{v''} = 13.55\ cm^{-1}$

COMMENT

→ $B_{v'} < B_{v''}$ and $B = \frac{h}{8\pi^2 \mu r_e^2 c}$. Therefore, r_e is decreasing.

Hence, $v' > v''$, i.e, the molecule is moving from lower vibration state to higher vibration state. This is possible for Vibrational-Rotational Spectra or Absorption Electronic Spectra.

→ Band Head is for m = +ve, i.e, R-Branch. R-Branch will show Band Degradation.

c) The Rotational Spectra is observed when molecule makes a transition from one Rotational level to another of the same vibrational state.

Rotational Energy Levels, $E = \frac{L^2}{2I} = \frac{J(J+1)h^2}{8\pi^2 I} = hcBJ(J+1)$

where $B = \frac{h}{8\pi^2 I c}$

Rotational Frequency, $\omega = \sqrt{\frac{2E}{I}}$

where $I = \mu r^2$ → μ = reduced mass of CO ⇒ $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

r = Internuclear Radius = 1.128 Å

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 = \left(\frac{12 \times 16}{12+16}\right) 1.67 \times 10^{-27} \times (1.128 Å)^2 \Rightarrow I = 1.457 \times 10^{-46}\ kg\,m^2$$

For $J = 1$,

$$E_1 = \frac{1 \times 2 \times h^2}{8\pi^2 I}$$

$$E_1 = 7.6323 \times 10^{-23}\ J$$

$$\omega_1 = \sqrt{\frac{2E_1}{I}}$$

$$\boxed{\omega_1 = 1.0235 \times 10^{12}\ rad\ s^{-1}}$$

ν is asked?

For $J = 10$,

$$E_{10} = \frac{10 \times 11 \times h^2}{8\pi^2 I}$$

$$E_{10} = 4.1977 \times 10^{-21}\ J$$

$$\omega_{10} = \sqrt{\frac{2E_{10}}{I}}$$

$$\boxed{\omega_{10} = 7.5906 \times 10^{12}\ rad\ s^{-1}}$$

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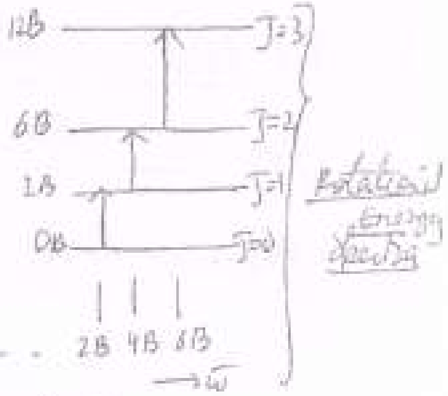
Rotational Spectra, $F(J) = \frac{E(J)}{hc} = BJ(J+1)$

(energy in wave number)

$$\bar{\omega} = F(J+1) - F(J)$$

$$\bar{\omega} = 2B(J+1) = 2B, 4B, 6B \ldots$$



Therefore, Rotational Spectra has constant spacing of 2B

$$2B = \frac{2h}{8\pi^2 I c} \Rightarrow \boxed{2B = 3.8395\ cm^{-1}}$$

This is the separation.

Due to low energy of Rotational Levels, the Spectra is in Microwave Region & called so.

Q5. (a) Q-Value of the Reaction can be defined as the Energy equivalent of the ~~Mass~~ Difference in the Mass of Reactants and the mass of the Products. It can also be defined as the Kinetic Energy of Products minus Kinetic Energy of Reactants.

$$a + A \longrightarrow b + B$$

$$m_a c^2 + T_a + m_A c^2 + T_A = m_b c^2 + T_b + m_B c^2 + T_B$$

$$Q = [(m_a + m_A) - (m_b + m_B)]c^2 = (T_b + T_B) - (T_a + T_A)$$

Numerical → $^{9}_{4}Be + ^{4}_{2}He \rightarrow ^{12}_{6}C + ^{1}_{0}n$

$$Q = [m(^9Be) + m(^4He) - m(^{12}C) - m(^1n)]c^2$$

Given → $m(^9Be) = 9.012183\,u$

$m(^4He) = 4.002603\,u$

$m(^{12}C) = 12.000\,u$

we know → $m(^1_0n) = 1.0086652\,u$

$$Q = 6.1208 \times 10^{-3}\, uc^2$$

$$Q = 9.1996 \times 10^{-13}\, J$$

$$\boxed{Q = 5.7497\ MeV}$$

This is the Required Q-Value. as $\boxed{Q > 0}$, it is an exothermic reaction and results in emission of energy

6/10

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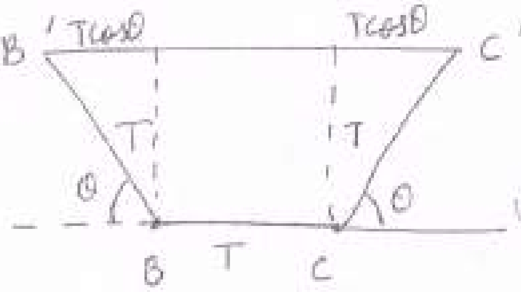
b) Let rotation by θ be a symmetric rotational operation. Since rotation by 2π is always results in same lattice structure, θ must be an integral multiple of 2π.

$$\boxed{2\pi = n\theta} \quad \text{—(1)}$$



When B and C are rotated by θ, we get B' & C'. As, θ is an symmetric operation, they must be obtained by Lattice Translation vector, T. From the figure, $B'C' = m(BC)$ where m is an integer

$$2T\cos\theta + T = mT$$

$$\boxed{\cos\theta = \frac{m-1}{2}} \quad \text{—(2)}$$

as $\cos\theta$ is limited between $[-1, 1]$.

$m = -1, 0, 1, 2, 3$

$\theta = 180°, 120°, 90°, 60°, 0°/360°$ (from 2)

from 1, $\boxed{n = 0, 1, 2, 3, 4, 6}$

Therefore, only these rotational symmetry are allowed. Rotational for $n = 5, 7, 8, 9\ldots$ so on & are not permissible.

Hence, any arbitrary rotation axis is not permitted in a crystal lattice.

c) There are 6 types of quarks and their corresponding antiquarks ~~set~~ in 3-generation

	Isospin Comp I_Z	Hypercharge, Y $Y = S + B$	Strangeness, ~~Baryon Number~~ S
up Quark, u	$+1/2$	$1/3$	0
Down Quark, d	$-1/2$	$1/3$	0
Strange Quark, s	0	$-2/3$	-1
$\bar{u}$	$-1/2$	$-1/3$	0
$\bar{d}$	$+1/2$	$-1/3$	0
$\bar{s}$	0	$2/3$	$+1$

★ all quarks have Baryon Number $\frac{-1}{3}$ for antiquarks, $B = -\frac{1}{3}$

According to Quark Model of Hadrons, all Baryons are made of Three Quarks and all Mesons are made of Quark & Anti-Quark.

PROTON, p = uud

$$q = \frac{2e}{3} + \frac{2e}{3} - \frac{e}{3} = e \checkmark$$

$$S = 0 + 0 + 0 = 0 \checkmark$$

$$B = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \checkmark$$

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NEUTRON, n = udd

$$q = \frac{2e}{3} - \frac{e}{3} - \frac{e}{3} = 0 \checkmark$$

$$S = 0 + 0 + 0 = 0 \checkmark$$

$$B = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \checkmark$$

d) Given Logical Expression, $Y = \left[A\bar{B}(C+BD) + \bar{A}\bar{B}\right]C$

$Y = \left[A\bar{B}C + A\bar{B}BD + \bar{A}\bar{B}\right]C$

$Y = \left[A\bar{B}C + \bar{A}\bar{B}\right]C$ → $X.\bar{X} = 0$

→ $0 + X = X$

$Y = \left[A\bar{B}CC + \bar{A}\bar{B}C\right]$ → $X.X = X$

$Y = A\bar{B}C + \bar{A}\bar{B}C$

$Y = \bar{B}C(A+\bar{A})$ → $X+\bar{X} = 1$

→ $X+1 = 1$

$Y = \bar{B}C$ → $X.1 = X$

This is the required simplified Logical Expression.

→ More complex expressions can be solved using the method of Karnaugh Map which simplify the process.

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(e) A Transistor is a three-terminal semi-conductor device in which a thin layer which is lightly doped with one-type of dopant is sandwiched between two thick layers which are more heavily doped with opposite-type of dopant.

n-p-n

→ Emitter & Collector = n-type

→ Base = p-type

→ The Majority Charge carrier and responsible for Diffusion Current is electrons.

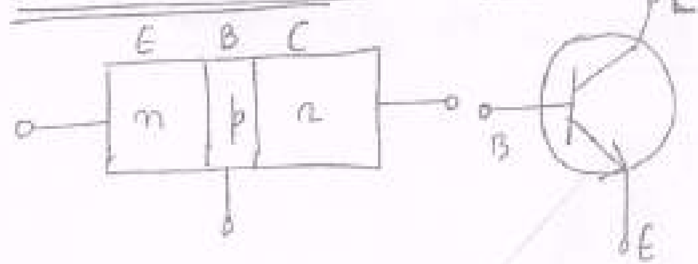
p-n-p

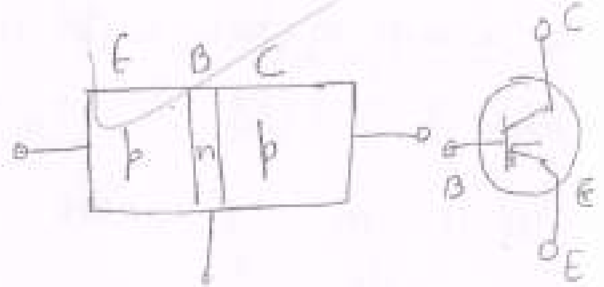
→ Emitter & Collector = p-type

→ Base = n-type

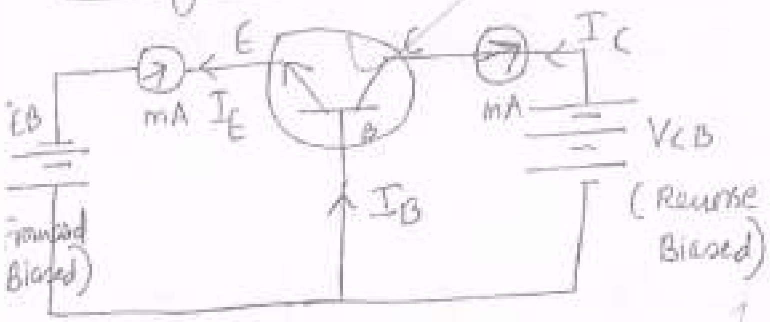
→ The Majority Charge Carrier & responsible for Diffusion Current is holes.

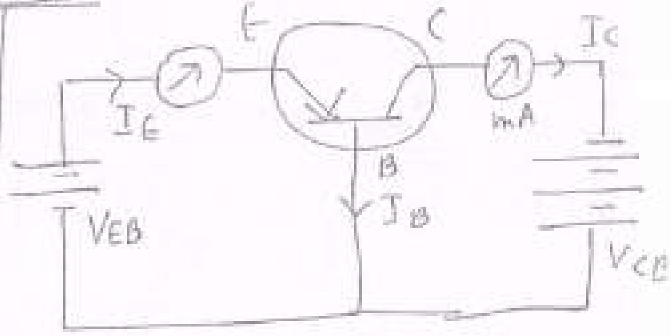
DEVICE STRUCTURE & SYMBOL





Biasing Circuits in Common Base Configuration





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Q8. According to the Debye Theory of Specific Heat of Solids,

→ The Solid comprises of atoms which act as coupled oscillators

→ A range of frequency can pass through them from 0 to ν_D (Debye frequency)

→ The energy is quantised according to Einstein's theory,

$$\boxed{\varepsilon = nh\nu}$$

For Density of States, the wave equation is $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

where $\psi = A \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \cos(2\pi \nu t)$

$$n_x^2 + n_y^2 + n_z^2 = R^2 = \left(\frac{2\nu L}{v_s}\right)^2$$

The Density of States, $D(\nu)$ is equal to the volume in the sphere of radius R to $R+dR$ in n_x, n_y, n_z space in the first quadrant.

$$D(\nu) = \frac{dn}{d\nu} = \frac{1}{8} 4\pi R^2 dR$$

$$\boxed{D(\nu) = \left(\frac{4\pi V}{v_s^3}\right) \nu^2 d\nu}$$

There are two transverse velocities & one longitudinal. Therefore,

$$\boxed{D(\nu) = (4\pi V)\left(\frac{2}{v_t^3} + \frac{1}{v_l^3}\right)\nu^2 d\nu} \quad \text{—(1)}$$

According to the Law of Equipartition of energy, N - 3-dimensional oscillators is equal to $3N$ - 1-dimensional oscillator.

$$3N = \int_0^{\nu_D} D(\nu)\, d\nu = \int_0^{\nu_D} 4\pi V \left(\frac{2}{V_t^3} + \frac{1}{V_l^3}\right)\nu^2 d\nu$$

$$\boxed{3N = \frac{4\pi V}{3}\left(\frac{2}{V_t^3} + \frac{1}{V_l^3}\right)\nu_D^3} \quad \text{—(2)}$$

The Total Energy, $\mathcal{E} = \int_0^{\nu} \bar{\varepsilon}\, D(\nu)\, d\nu$

According to Einstein Theory, $\bar{\varepsilon} = \dfrac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$

$$\mathcal{E} = \int_0^{\nu_D} \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\, 4\pi V\left(\frac{2}{v_t^3} + \frac{1}{v_l^3}\right) \nu^2 d\nu \qquad \text{— from (1)}$$

$$\mathcal{E} = \int_0^{\nu_D} \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\, \frac{9N}{\nu_D^3} \nu^2 d\nu \qquad \text{— from (2)}$$

Let $x = \dfrac{h\nu}{k_B T}$ and $\Theta_D = \dfrac{h\nu_D}{k_B}$, i.e., Debye Temperature. Then

$$\boxed{\mathcal{E} = 9Nk_B \frac{T^4}{\Theta_D^3} \int_0^{\Theta_D/T} \frac{x^3\, dx}{e^x - 1}}$$

→ High Temperature Limit, $x \ll 1$, i.e., $e^x - 1 \simeq x$

$$\mathcal{E} = 9Nk_B \frac{T^4}{\Theta_D^3} \int_0^{\Theta_D/T} x^2 dx = 9Nk_B \frac{T^4}{\Theta_D^3} \left(\frac{\Theta_D}{T}\right)^3 \frac{1}{3}$$

($\int x^2 dx = \frac{x^3}{3}$)

$$\boxed{\mathcal{E} = 3Nk_B T \quad \text{and} \quad C_V = \frac{\partial E}{\partial T} = 3Nk_B}$$

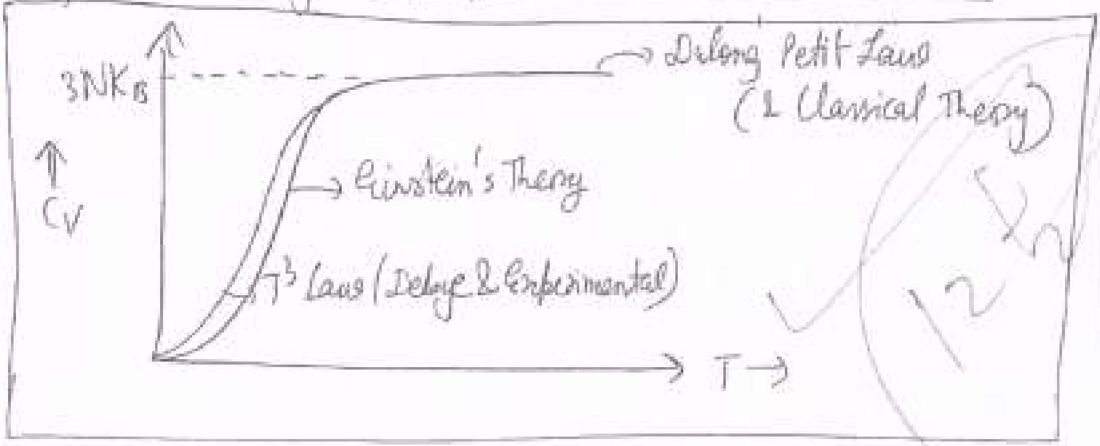
This is the Dulong Petit's Law.

→ Low Temperature Limit, $x \gg 1$, i.e., $e^x - 1 \approx e^x$ & $\frac{\Theta_D}{T} \to \infty$

$$\mathcal{E} = 9Nk_B \frac{T^4}{\Theta_D^3} \int_0^{\infty} x^3 e^{-x} dx = 9Nk_B \frac{T^4}{\Theta_D^3} \left(\frac{\pi^4}{15}\right)$$

$$\boxed{\mathcal{E} = \frac{3Nk_B \pi^4}{5} \frac{T^4}{\Theta_D^3} \quad \text{and} \quad C_V = \frac{\partial E}{\partial T} = \frac{12}{5}\pi^4 Nk_B \left(\frac{T}{\Theta_D}\right)^3}$$

Therefore, the Debye Theory appropriately proves the T^3 Law which is followed by Specific Heat at Low Temperature.



b) The Potential observed by the electrons in the lattice according to the Band Theory of Solids as developed by Kronig Penney Model is –

$$V(x) = \begin{cases} 0 , & 0 < x < a \\ V_0 , & -b < x < 0 \end{cases}$$

The time - independent Schrodinger Equation is –

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

in region I, $0 < x < a$, $V(x) = 0$ $\boxed{\frac{d^2\psi_1(x)}{dx^2} + \alpha^2\psi_1(x) = 0}$ where $\alpha^2 = \frac{2mE}{\hbar^2}$

in region II, $-b < x < 0$, $V(x) = V_0$ $\boxed{\frac{d^2\psi_2(x)}{dx^2} - \beta^2\psi_2(x) = 0}$ where $\beta^2 = \frac{2m(V_0 - E)}{\hbar^2}$

As the Potential is Periodic, according to the Bloch's Theorem,

$$\boxed{\psi(x) = \exp(ikx)u(x)}$$

using this above,

$$\frac{d^2u_1(x)}{dx^2} + 2ik\frac{du_1(x)}{dx^2} + (\alpha^2 + k^2)u_1(x) = 0$$

$$\frac{d^2u_2(x)}{dx^2} + 2ik\frac{u_2(x)}{dx^2} - (\beta^2+k^2)u_2(x) = 0$$

The solution of the above differential equation is –

$$u_1(x) = A\exp[i(\alpha-k)x] + B\exp[-i(\alpha+k)x]$$

$$u_2(x) = C\exp[(\beta-ik)x] + D\exp[-(\beta+ik)x]$$

According to Born's Interpretation, the wave function & its derivative must be continuous for all x.

$$u_1(0) = u_2(0), \qquad u_1(a) = u_2(-b)$$

$$\left.\frac{du_1(x)}{dx}\right|_0 = \left.\frac{du_2(x)}{dx}\right|_0, \qquad \left.\frac{du_1(x)}{dx}\right|_a = \left.\frac{du_2(x)}{dx}\right|_{-b}$$

The Non-Trivial Solution of above equations is

$$\left(\frac{\beta^2-\alpha^2}{2\alpha\beta}\right)\sin\alpha a\sin\beta b + \cosh\beta b\cos\alpha a = \cos k(a+b)$$

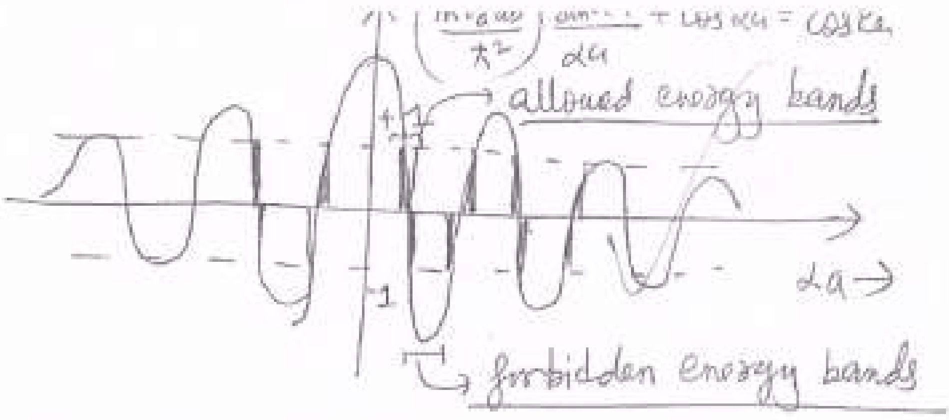
considering the limit, $V_0 \to \infty$ & $b \to 0$ but $V_0 b$ is finite, ie, the potential is in the form of Delta function

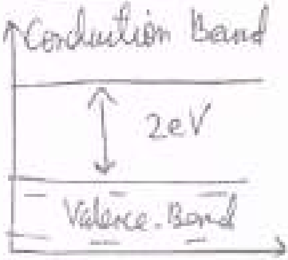
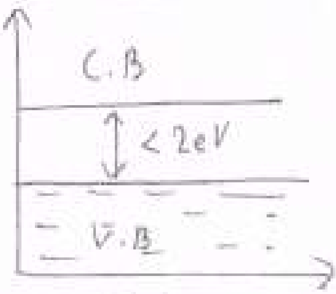
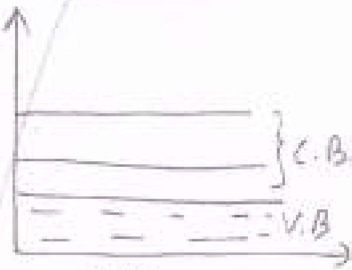
$$\boxed{\left(\frac{mV_0 ab}{\hbar^2}\right)\left(\frac{\sin\alpha a}{\alpha a}\right) + \cos\alpha a = \cos ka}$$

This is the required relation which relates K Vector and Energy as $\left(\alpha^2 = \frac{2mE}{\hbar^2}\right)$

The R.H.S. is limited between (–1 & +1). Therefore, only those values of α & hence energy are allowed which ensure that L.H.S. is between +1 & –1.

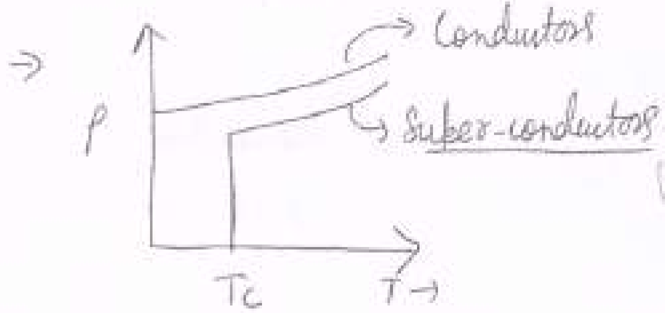
This gives rise to allowed energy bands & forbidden energy bands.



Insulators	Semi-Conductors	Conductors
		
eg. Wood, glass	eg. Si, Ge	eg. Copper, Iron
→ The Valence Band is completely filled. Therefore effective number of e^- is zero. Hence, very poor conductivity.	→ The effective number of e^- is zero as V.B. is completely filled.	→ The V.B. is partially filled. Hence, large number of effective electrons & hence, high conductivity
→ As the forbidden energy gap is large, the conductivity doesn't increase much with temperature.	→ But Energy gap is small & conductivity increases with temperature. This gives them wide applications.	

(c) Superconductor can be defined as a substance which exhibits zero resistivity or infinite conductivity below a certain critical temperature.

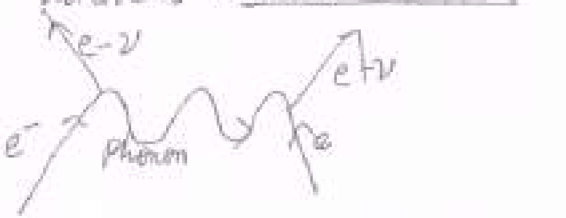
Superconductors	Conductors
→ (graph below)	→ Resistivity is small but finite even at 0K.
Resistivity (ρ) is zero below T_c.	→ for $\rho = 0$, $E = 0$; $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$. Therefore, B (magnetic field) is constant.
→ Show Meisner effect, ie, complete & sudden expulsion of magnetic field below T_c.	
→ The current is a surface current.	→ The current is in the Body.



Cooper Pairs

It was given by the BCS Theory. According to it, Cooper Pairs is a combined state of Two electrons which are bound together by small energy & are responsible for ~~the~~ super-conductivity.

When an electron moves in a lattice it creates distortions. An electron passing through the region experiences a force of attraction. This overcomes even coulombic repulsion as it is immediate whereas Lattice Distortion is delayed. These e⁻'s are cooper pairs.



Q1(a) The Probability Current Density is ~~the Mass~~

$$J = \text{Re}\left[\psi^* \frac{\hbar}{im}\frac{d\psi}{dx}\right]$$

given: $\psi(x,t) = 2\exp i(kx - \omega t)$

$$J = \text{Re}\left[2e^{-i(kx-\omega t)} \frac{\hbar}{\cancel{i}m}(\cancel{i}k)2e^{i(kx-\omega t)}\right]$$

$$J = \frac{4\hbar k}{m} = \frac{4p}{m} = 4v$$

[Momentum, $p = \hbar k$]

$$\boxed{J = 4v}$$

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Therefore, the Probability Current Density is directly ~~current dens~~ proportional to the Velocity. This is expected as J gives the rates at which Probability associated with the Particle with flowing. for a Plane Wave, it will be equal to the Velocity of the Waves.

b) The ESR & NMR technique utilise the phenomenon of splitting of energy levels of electron & nucleus respectively in the presence of external magnetic field. When the frequency of second magnetic field is equal to precessional frequency, there is transition from one state to another. This is called Resonance.

A) The ESR & NMR frequency lie in the Radio frequency range.

ESR Microwave

-1

The Electron Spin Resonance

$$\Delta E = g\mu_B B$$

at resonance, $$\nu = \frac{g\mu_B B}{h}$$

Therefore, Lande's g-factor, $$g = \frac{h\nu}{\mu_B B}$$

$4\frac{1}{2}$

6

/10

Given $B = 9.5 \times 10^{3}$ Tesla

$\nu = 340$ kHz

$\mu_B = \frac{eh}{4\pi m_e}$

$$g = 2.5579 \times 10^{-9}$$

This is approximately 0. So, electron is not in $l = 0$, s-state.

(c) ~~$\psi(x)$~~ $$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ and $\Delta p_x = \sqrt{\langle p_x\rangle^2 - \langle p_x\rangle^2}$

$$\langle x\rangle = \int_{-\infty}^{+\infty} x|\psi(x)|^2 dx = \int_{-\infty}^{+\infty} x\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx = 0$$ [as it is an odd function $f(x) = -f(x)$]

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2|\psi(x)|^2 dx = \int_{-\infty}^{+\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} x^2 \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx$$

$$\langle x^2\rangle = 2\int_0^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} x^2 \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx$$

$$\langle x^2\rangle = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{\Gamma(3/2)}{2\left(\frac{m\omega}{\hbar}\right)^{3/2}}\right]$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{(n+1)/2}}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega} \qquad \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle p_x\rangle = \int_{-\infty}^{+\infty}\psi^*(x) - i\hbar\frac{\partial\psi(x)}{\partial x} dx = \int_{-\infty}^{+\infty}\left(\frac{m\omega}{\hbar\pi}\right)^{1/2} e^{-\frac{m\omega x^2}{2\hbar}} - i\hbar\left(-\frac{m\omega x}{\hbar}\right) e^{-\frac{m\omega x^2}{2\hbar}} dx$$

$$\langle p_x\rangle = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} i\hbar\left(\frac{m\omega}{\hbar}\right)\int_{-\infty}^{+\infty} x e^{-m\omega x^2/\hbar} dx = 0$$ [as integral is an odd function]

$$\langle p_x^2\rangle = \int_{-\infty}^{+\infty}\psi^*(x) - \hbar^2\frac{d^2}{dx^2}\psi(x) dx = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}(-\hbar^2)\int_{-\infty}^{+\infty} e^{-\frac{m\omega x^2}{2\hbar}}\frac{d}{dx}\left[\left(-\frac{m\omega x}{\hbar}\right) e^{-\frac{m\omega x^2}{2\hbar}}\right] dx$$

$$\langle p_x^2\rangle = \left(\frac{m\omega}{\hbar\pi}\right)\hbar^2\left(\frac{m\omega}{\hbar}\right)\int_{-\infty}^{+\infty} e^{-\frac{m\omega x^2}{2\hbar}}\frac{d}{dx}\left[x e^{-\frac{m\omega x^2}{2\hbar}}\right] dx$$

Not Asked

where did you show zeropoint energy ~ 10

Similarly 11

(d) In Quantum Mechanics every angular momentum operator satisfy the following commutation relation.

Commutator $[A, B] = AB - BA$

$[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$

$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

i) $[L_z, X] = [XP_y - YP_x, X]$

$= [XP_y, X] - [YP_x, X]$ $\quad [AB, C] = A[B, C] + [A, C]B$

$= X[P_y, X] + [X, X]P_y - Y[P_x, X] - [Y, X]P_x$

$[X,X] = 0$ multiplication type; $[Y,X] = 0$ (multiplication type)

$[X, P_x] = [Y, P_y] = [Z, P_z] = i\hbar$ (all other are zero).

$\boxed{[L_z, X] = -Y(-i\hbar)} \Rightarrow \boxed{[L_z, X] = i\hbar Y}$

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ii) $[L_z, Y] = [XP_y - YP_x, Y]$

$= [XP_y, Y] - [YP_x, Y]$

$= X[P_y, Y] + [X, Y]P_y - Y[P_x, Y] - [Y, Y]P_x$

$[X,Y] = 0$, $[P_x,Y] = 0$, $[Y,Y] = 0$ from above commutation relation

$= X(-i\hbar)$

$\boxed{[L_z, Y] = -i\hbar X}$

(c) To prove, $\exp i\theta(\vec{\sigma}\cdot\hat{n}) = \cos\theta + i(\vec{\sigma}\cdot\hat{n})\sin\theta$

where $\vec{\sigma}$ = Pauli Vector

$\vec{\sigma} = \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We know $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\exp i\theta(\vec{\sigma}\cdot\hat{n}) = \sum_{n=0}^{\infty} \frac{[i\theta(\vec{\sigma}\cdot\hat{n})]^n}{n!}$$

$$= \sum_{p=0}^{\infty} \frac{[i\theta(\vec{\sigma}\cdot\hat{n})]^{2p}}{(2p)!} + \sum_{q=0}^{\infty} \frac{[i\theta(\vec{\sigma}\cdot\hat{n})]^{2q+1}}{(2q+1)!}$$

$i^{2p} = (-1)^{p}$ and $i^{2q+1} = i(-1)^{q}$

Now we know $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = I(\vec{A}\cdot\vec{B}) + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

If $(\vec{A} = \vec{B}) = \hat{n}$ $\quad (\vec{\sigma}\cdot\hat{n})^2 = I(1) + i\vec{\sigma}\cdot(0)$

$(\vec{\sigma}\cdot\hat{n})^2 = I$

$(\vec{\sigma}\cdot\hat{n})^{2p} = I$ and $(\vec{\sigma}\cdot\hat{n})^{2q+1} = \vec{\sigma}\cdot\hat{n}$

using these above.

$$\exp i\theta(\vec{\sigma}\cdot\hat{n}) = \left[\sum_{p=0}^{\infty} \frac{(-1)^p\theta^{2p}}{2p!}\right] I + \left[\sum_{q=0}^{\infty} \frac{(-1)^q\theta^{2q+1}}{(2q+1)!}\right] i(\vec{\sigma}\cdot\hat{n})$$

$$\boxed{\exp i\theta(\vec{\sigma}\cdot\hat{n}) = I\cos\theta + i(\vec{\sigma}\cdot\hat{n})\sin\theta}$$

Hence proved

$$\sin\theta = \sum_{m=0}^{\infty} \frac{(-1)^m\theta^{2m+1}}{(2m+1)!}$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!}$$